Characterizing and Testing Principal Minor Equivalence of Matrices

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Abstract

Two matrices are said to be principal minor equivalent if they have equal corresponding principal minors of all orders. We give a characterization of principal minor equivalence and a deterministic polynomial time algorithm to check if two given matrices are principal minor equivalent. Earlier such results were known for certain special cases like symmetric matrices, skew-symmetric matrices with 0, 1, -1-entries, and matrices with no cuts (i.e., for any non-trivial partition of the indices, the top right block or the bottom left block must have rank more than 1).

As an immediate application, we get an algorithm to check if the determinantal point processes corresponding to two given kernel matrices (not necessarily symmetric) are the same. As another application, we give a deterministic polynomial-time test to check equality of two multivariate polynomials, each computed by a symbolic determinant with a rank 1 constraint on coefficient matrices.

1 Introduction

The determinant of a matrix is a fundamental object of study in mathematics that has found numerous applications throughout computer science, physics, and other fields. A *minor* of a matrix is the determinant of one of its square submatrices and its order is the size of the corresponding submatrix. A *principal minor* of a matrix is a minor obtained by deleting the same set of rows and columns. Principal minors play an important role in a variety of applications, for example, convexity of functions and positive semidefinite matrices [BV04], the linear complementarity problem and P-matrices [Mur72], counting number of forests via the Laplacian matrix [BS11], and inverse eigenvalue problems [Fri77].

In this paper, we investigate a basic question about principal minors – what is the relationship between two matrices which have equal corresponding principal minors of all orders (i.e., two matrices A and B such that for all $S \subseteq \{1, 2, \ldots, n\}$, $\det(A[S,S]) = \det(B[S,S])$). We call two such matrices to be *principal minor equivalent (PME)*. Observe that two matrices are PME if and only if all their corresponding principal submatrices have the same set of eigenvalues. We seek answers of the following two questions.

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Question 1 (characterization). Can we identify a property $\mathcal{P}$ such that two matrices are PME if and only if they satisfy $\mathcal{P}$?

Question 2 (efficient algorithm). Can we efficiently check whether two matrices are PME or not?

The question of characterizing the relationship between two PME matrices has been extensively studied [ES80, Loe86, Ahm23, BCCL21, BC16]. One motivation for studying this question comes from the problem of learning determinantal point processes [KT12, UBMR17, Bru18] and the closely related principal minor assignment problem [GT06, RKT15, BU24]. While our original motivation to study this question came from an application to the polynomial identity testing problem (see Section 1.1).

To move towards characterizing PME matrices, let us first consider some trivial operations which preserve all the principal minors. Two matrices A and B are called *diagonally similar* if there exists an invertible diagonal matrix D such that $A = DBD^{-1}$. We call two matrices A and B *diagonally equivalent* if A is diagonally similar to B or B^T. It is easy to verify that any two diagonally equivalent matrices are PME. Interestingly, Engel and Schneider [ES80] showed that the converse is also true when one of the matrices is symmetric. That is, principal minor equivalence of a symmetric matrix with another matrix implies their diagonal equivalence (in fact, diagonal similarity). As one can efficiently check whether two matrices are diagonally equivalent or not, it also yields an efficient algorithm to decide principal minor equivalence in this case.

In general, principal minor equivalence does not imply diagonal equivalence, as demonstrated by the following example. Consider the following block diagonal matrix A and a block upper triangular matrix B:

$$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}, \qquad B = \begin{pmatrix} A_1 & A_3 \\ 0 & A_2 \end{pmatrix}. \tag{1}$$

It is easy to see that A and B are principal minor equivalent oblivious to the entries of A_3, but they are not diagonally equivalent. Such matrices that can be written as a block upper triangular matrix by permuting some rows and corresponding columns are called *reducible* matrices (and irreducible otherwise). For any $n \times n$ matrix A, define a graph with the vertex set $[n]$ and allow an edge (i, j) if and only if the (i, j)-th entry of A is nonzero. We can equivalently define reducible matrices as the ones whose graph has more than one *strongly connected components*. One can show that two matrices are PME if and only if they have the same set of irreducible blocks and their corresponding irreducible blocks are PME (see, for example, [Ahm23, Section 5]). Hence, we can restrict our focus to irreducible matrices.

In a series of works, Hartfiel and Loewy [HL84], and Loewy [Loe86] extended the result of Engel and Schneider [ES80] to general irreducible matrices with no *cuts*. An $n \times n$ matrix A is said to have a cut $X \subseteq [n]$, if $2 \leq |X| \leq n - 2$ and both the submatrices $A[X, \overline{X}]$ and $A[\overline{X}, X]$ have rank one (the submatrices cannot have rank zero if A is irreducible). They showed that for any irreducible matrix A with no cuts and any matrix B, if A and B are PME, then A and B are also diagonally equivalent. So, the case which remained unclear was that of irreducible matrices with cuts. Engel and Schneider [ES80, Example 3.7] had given an example of two 4×4 matrices which are PME, but not diagonally equivalent. Clearly, both these matrices must have a cut.

The cut-transpose operation. Recently, Ahmadieh[Ahm23, Lemma 4.5] gave a general recipe that for any irreducible matrix A with a cut, finds another matrix B that is PME to A, but not necessarily diagonally equivalent to A. For this they define an operation on matrices with a cut, which we refer as cut-transpose. Consider a matrix A and let X be a cut of A. From the definition of a cut, A must be of the following form:

$$A = \begin{pmatrix} M & pq^T \\ uv^T & N \end{pmatrix},$$

where the submatrix $A(X, X) = M$ and $A(\overline{X}, \overline{X}) = N$ and p, q, u, v are column vectors of appropriate dimensions. Define a *cut-transpose* operation on A with respect to cut X, which transforms A to a new matrix $\mathrm{ct}(A, X)$ as follows:

$$\mathrm{ct}(A, X) = \begin{pmatrix} M & pu^T \\ qv^T & N^T \end{pmatrix}.$$

Ahmadieh [Ahm23] showed that cut-transpose is a principal minor preserving operation. A natural conjecture would be that any two irreducible PME matrices are related by a sequence of cut-transpose operations. To elaborate, let us define any two matrices A and B as *cut-transpose equivalent* if there is a sequence $A = A_0, A_1, \ldots, A_k$ of matrices such that for each $0 \leq i \leq k-1$, $A_{i+1} = \mathrm{ct}(A_i, X_i)$ for some cut X_i of A_i, and A_k is diagonally equivalent to B. Can one show that two irreducible matrices are PME if and only if they are cut-transpose equivalent?

Interestingly, Boussaïri and Chergui [BC16] had shown precisely this for a special case, when the two matrices are skew-symmetric with entries from $\{-1, 0, 1\}$ and all their off-diagonal entries in the first row are nonzero. Moreover, they conjectured that it should be true for arbitrary skew-symmetric matrices. In a follow up work, Boussaïri, Chaïchaâ, Chergui, and Lakhlifi [BCCL21] proved the same for another special case called generalized tournament matrices (non-negative matrices A with $A + A^T = J_n - I_n$, where J_n is all ones matrix). The cut-transpose operation in their settings is called as HL-clan-reversal or clan-inversion. Both these work build on a combinatorial result [BILT04] about directed graphs with a similar flavor. The combinatorial result, in turn, is a generalization of Gallai's theorem [Gal67] which states that if two partially ordered sets have the same comparability graph, then they are related by a sequence of orientation reversal operations (see [BILT04, M85̈]). This orientation reversal on a poset is a special instance of cut-transpose on the corresponding skew-symmetric matrix.

This series of works strengthens the confidence in the conjecture that cut-transpose equivalence should be a characterization of PME for arbitrary irreducible matrices. However, their techniques are graph-theoretic and it is not clear how they can be generalized to arbitrary matrices. We instead employ algebraic techniques and show that conjecture is indeed true, thereby completely resolving Question 1. This extends the above results and also proves the conjecture of Boussaïri and Chergui [BC16] about skew-symmetric matrices. Moreover, we show that for any two $n \times n$ irreducible PME matrices A and B, the cut-transpose sequence contains at most $2n$ matrices.

Theorem 1.1. *Let A and B be two $n \times n$ irreducible matrices over any field. Then, A and B are principal minor equivalent if and only if there exists a sequence of $n \times n$ matrices $(A = A_0, A_1, \ldots, A_k)$ with $k < 2n$ such that*

$$\text{for } 0 \leq i \leq k-1,\ A_{i+1} = \mathrm{ct}(A_i, X_i) \text{ for some cut } X_i \text{ of } A_i \tag{2}$$

and A_k is diagonally equivalent to B.

Now, let us come to the question of an efficient algorithm to check if two given matrices are PME (Question 2). If one is allowed the use of randomness, then there is a simple algorithm for this task via a reduction to polynomial identity testing. Consider a $n \times n$ diagonal matrix Y with variables $y_1, y_2, \ldots, y_n$ in the diagonal. Observe that two $n \times n$ matrices A and B are PME if and only if the following is a polynomial identity (i.e., coefficient-wise equality)

$$\det(A + Y) = \det(B + Y).$$

There is a simple randomized algorithm for polynomial identity testing: just plug-in some random numbers for the variables and then check the equality (see [Sch80, DL78, Zip79]). There is no deterministic polynomial time algorithm known for polynomial identity testing in general, but we can still ask if there is one for this special case. We answer this question positively. Recall the earlier discussion about reducible matrices and note that testing PME for two matrices reduces to the same for their corresponding irreducible blocks.

Theorem 1.2. *There exists a deterministic polynomial-time algorithm that for any two given $n \times n$ matrices A and B over any field, decides whether the corresponding principal minors of A and B are equal or not. If they are equal, then as a certificate, the algorithm outputs cut-transpose sequences for the corresponding irreducible blocks of the two matrices as guaranteed by Theorem 1.1.*

1.1 Applications

Polynomial Identity Testing. As mentioned earlier our motivation for the principal minor equivalence problem came from the polynomial identity testing (PIT) problem. Given two multivariate polynomials in a succinct representation, the PIT problem asks to decide whether the two polynomials are identical (i.e., all corresponding coefficients are equal). One of the widely studied and useful representation for multivariate polynomials is the *determinantal representation*. We say that a polynomial $f(x_1, \ldots, x_m) \in \mathbb{F}[x_1, \ldots, x_m]$ has a determinantal representation of size n if there exists matrices $A_0, A_1, \ldots, A_m \in \mathbb{F}^{n \times n}$ such that $f = \det(A_0 + \sum A_i x_i)$. The determinantal representation is known to be almost as expressive as algebraic circuits (see [Val79] for more details). The PIT problem admits a randomized polynomial-time algorithm [Sch80, DL78, Zip79]. Obtaining a deterministic algorithm for PIT remains a challenging open problem that would have interesting implications in proving lower bounds, and many other algorithmic applications (see, for example, [SY10]). Unable to solve it for the general setting, the problem has been studied for various restricted settings.

One such restricted setting is symbolic determinant under rank one restriction. Here we ask for testing whether $\det(A_0 + \sum_{i=1}^{m} A_i x_i) = 0$, for given matrices A_i, where $\text{rank}(A_i) = 1$ for $1 \leq i \leq m$. There has been a lot of interest in this particular setting because of its connections with some combinatorial optimization problems like bipartite matching and linear matroid intersection (see [Edm67, Lov89, NSV92]), and algebraic problems like maximum rank matrix completion (see [IKS10, Gee99, Mur93]). The connection with combinatorics also gives a deterministic polynomial time algorithm for identity testing in this setting. In fact, there is also an efficient blackbox PIT (quasi-polynomial time) known for this case [GT17] (blackbox means that the algorithm cannot see the input, it can only evaluate the given polynomial at any point).

When we have an efficient algorithm to test whether a given polynomial from a class is zero, the next natural question one can ask is to test whether two given polynomials from that class are equal. If the class of polynomials is closed under addition, the equality question easily reduces to testing zeroness of a given polynomial (from the same class). Many well studied classes of polynomials have this property, for example, sparse polynomials, bounded-depth circuits, constant fan-in depth-3 circuits etc. On the other hand, there are classes like ROABPs, which are not closed under addition [KNS20], and for which the equality testing question has been studied independently [GKST16]. Symbolic determinant with rank one restriction is another such class. To the best of our knowledge, the class is not known to be closed under addition. Given that zeroness testing is known for this class, a natural extension would be to ask if two given polynomials from this class are equal. To the best of our knowledge, no non-trivial (deterministic) algorithm was known for testing equality of two polynomials from this class (symbolic determinant with rank one restriction). We show that this problem reduces to testing principal minor equivalence, and hence, has a deterministic polynomial-time algorithm.

Theorem 1.3. *There exists a deterministic polynomial time algorithm such that given two sequences of $n \times n$ matrices $(A_0, A_1, \ldots, A_m)$ and $(B_0, B_1, \ldots, B_m)$ over any field, with the rank of A_i and B_i being* *at most* 1 *for* $1 \leq i \leq n$, *it decides whether* $\det(A_0 + A_1y_1 + \ldots + A_my_m) = \det(B_0 + B_1y_1 + \ldots + B_my_m)$.

Determinantal Point Processes. As mentioned earlier, one motivation to study principal minors come from determinantal point processes (DPP). DPP are a family of probabilistic models which originated in physics [Mac75], and which has subsequently found a wide range of applications in machine learning [KT12], for example, document summarization, recommender systems, information retrieval etc. (see references given in [GBDK19, UBMR17]). Conventionally, a DPP is defined using principal minors of an $n \times n$ symmetric positive semidefinite matrix K, called a kernel, whose eigenvalues are between 0 and 1. The DPP corresponding to kernel matrix K is a probability distribution on subsets Y of $\{1, 2, \ldots, n\}$ such that for any subset $J \subseteq \{1, 2, \ldots, n\}$,

$$\Pr[J \subseteq Y] = \det(K_J),$$

where K_J is the principal minor of K corresponding to set J (see [Kul12]). DPPs are useful in settings where one needs to generate a diverse set of objects (larger principal minor means the vectors associated with the subset span a larger volume).

Symmetric DPPs (as defined above with a symmetric kernel matrix) have a significant expressive power, however they come with a limitation. Symmetric DPPs can model only repulsive interactions. That is, any pair of items has a negative correlation – selection of one item reduces the chances of selection of another item. To overcome this limitation, nonsymmetric determinantal point process has been proposed, that is, DPP with a nonsymmetric kernel matrix K. A nonsymmetric kernel matrix can model both positive and negative correlations. Lately, there have been a few works on nonsymmetric DPPs [Bru18, GBDK19, RRS$^+$22, HGDK22, Arn24]. One of the crucial questions in the study of DPPs is to understand how are two kernel matrices related which produce the same DPP, which was explicitly asked in some works on learning DPPs [Bru18, BU24]. This is precisely the principal minor equivalence problem. While it was already understood in the case of symmetric DPPs, we answer it for nonsymmetric DPPs in this work. Theoerm 1.1 gives a characterization of the set of matrices K' such that $DPP(K') = DPP(K)$ for a given kernel matrix K (not necessarily symmetric). Theorem 1.2 gives a deterministic polynomial time algorithm to test whether two given kernel matrices will produce the same DPP.

1.2 Proof overview

In this subsection, we give a short overview of the proof techniques involved in proving Theorems 1.1, 1.2 and 1.3. We start with Theorem 1.1 which characterizes principal minor equivalence of irreducible matrices by cut-transpose equivalence. We have already discussed that cut-transpose equivalence implies principal minor equivalence. Thus, only the other direction remains to be shown, i.e., principal minor equivalence implies cut-transpose equivalence.

Reduction to the case of all nonzero entries. Our proof of Theorem 1.1 works with an assumption that the matrices have all nonzero entries. We reduce the general case to this case using a technique from earlier works [Loe86, HL84, Ahm23], namely, the transformation $A \mapsto (A+Z)^{\text{adj}}$ (or $(A+Z)^{-1}$), where Z is a diagonal matrix with entries as distinct algebraically independent elements (or indeterminates). They showed that for any irreducible matrix A, the matrix $(A+Z)^{\text{adj}}$ has all nonzero entries. Moreover, two irreducible matrices A and B are PME if and only if $(A+Z)^{\text{adj}}$ and $(B+Z)^{\text{adj}}$ are. They have also shown that A and $(A+Z)^{\text{adj}}$ have the same set of cuts. In Lemma 2.15, we show that the cut-transpose operation commutes with operation $A \mapsto (A+Z)^{\text{adj}}$. This means that matrices A and B are cut-transpose equivalent if and only if $(A+Z)^{\text{adj}}$ and $(B+Z)^{\text{adj}}$ are. Hence, it is sufficient to prove Theorem 1.1 for matrices with all nonzero entries.

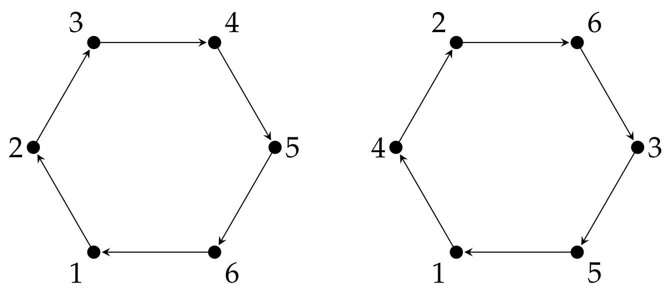

Figure 1: Directed graphs associated with two matrices A and B.

No common cuts. To prove the characterization for matrices with cuts, a natural strategy would be to somehow decompose the matrices along a chosen cut and then argue inductively for the obtained smaller pairs of matrices. From Loewy's characterization [Loe86], it follows that for any two irreducible PME matrices A and B, if A has a cut, then B must also have one. However, it is not necessary a subset of indices which is a cut in matrix A, is also a cut in matrix B. In fact, it is possible that the two matrices do not have even one cut in common. Following is such an example of two irreducible PME matrices.

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

To see that the two matrices A and B are PME, recall that the determinant is a sum over cycle covers in the directed graph associated with the matrix. And observe that the associated directed graphs (Figure 1) have only one cycle, and thus, both matrices have only one nonzero principal minor (i.e., $\det(A) = \det(B) = 1$). Now, to see that the two matrices do not have a common cut, observe that for these matrices, any cut corresponds to a path in associated directed graph. And there is no subset of vertices simultaneously forming a path in both the graphs.

We handle such cases with no common cuts by transforming one of the matrices to have a common cut with the other. Then we prove cut-transpose equivalence by induction based on the size of the matrices. The following points summarize our proof strategy.

1. For any two irreducible PME matrices A and B, we show that the matrix A has a cut in common either with matrix B or with another matrix B' obtained from B via a cut-transpose operation.

2. Then assuming that the two given matrices have a common cut, we "decompose" each matrix along a common cut to obtain two smaller matrices. We argue that the two obtained pairs of matrices are also PME and hence, are cut-transpose equivalent by induction hypothesis. Then we are able to lift their cut-transpose equivalence to the given matrices.

3. The base case for the induction is 4×4 matrices.

We now elaborate on each of the above points.

Base case: 4×4 matrices If A and B are 4×4 irreducible PME matrices, then we show (Lemma 3.2) that (i) either the two are diagonally equivalent or (ii) they have a common cut and

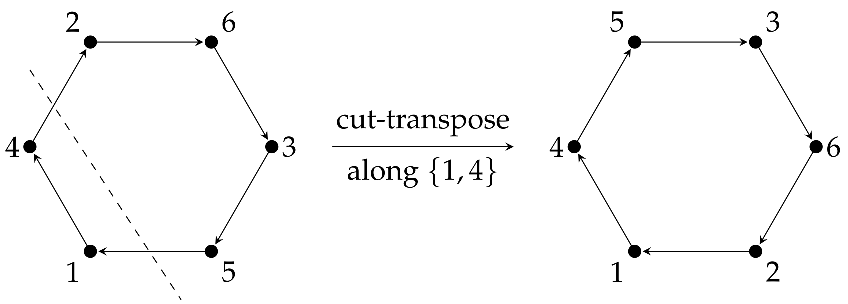

Figure 2: Applying cut-transpose on the directed graph associated with matrix B.

when we do a cut-transpose on matrix A along the common cut, we get a matrix diagonally equivalent to B. For 3×3 or smaller matrices, there is no cut, and hence the two matrices must be diagonally equivalent [HL84].

Getting a common cut. To get a common cut in the given matrices A and B, we consider a (inclusion-wise) minimal cut S in A. We show that if S is not a cut in B then S must have size two (Lemma 3.6). Moreover, in that case we can argue that there is a cut X in matrix B such that S is a cut in another matrix B', which is obtained by applying cut-transpose on B along cut X (Lemma 3.3). Clearly, proving cut-transpose equivalence between A and B' will imply the same between A and B. The proofs of these two lemmas build on some other technical claims (Theorem 3.1, Lemmas 3.2, 3.4, 3.5), and this is where most of the technical novelty lies.

Let us see how the matrix B' is obtained in the example described above. Observe that the matrix A has a cut $S = \{1, 2\}$, which is not a cut in matrix B. Let us consider the cut $X = \{1, 4\}$ in matrix B and apply cut-transpose along it. We obtain the following matrix B', which has $S = \{1, 2\}$ as a cut, as desired. Figure 2 shows the cut-transpose operation on the associated directed graph.

$$B = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \xrightarrow[\text{along } \{1,4\}]{\text{cut-transpose}} B' = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Decomposition into smaller matrices. One of our crucial ideas is to define the right decomposition of a matrix along one of its cuts. For an $n \times n$ matrix A with a cut $S \subseteq [n]$, we consider a decomposition of A into two matrices A_1 and A_2 defined as follows: choose two arbitrary indices $s \in \overline{S}$ and $t \in S$, and define $A_1 := A[S + s]$ and $A_2[\overline{S} + t]$. Recall that we assume all off-diagonal entries to be nonzero, hence, the choice of s and t do not really matter. As discussed earlier, we can assume that there is set S, which is a cut in both the matrices A and B. It is easy to see that if A and B are PME, then so are A_i and B_i, for $i = 1, 2$.

If S is a minimal cut of A, then we show that A_1 has no cut (Lemma 3.4). In that case, B_1 also does not have a cut and is diagonally similar to A_1 or A_1^T (from Loewy's characterization). If it so happens that A_2 and B_2 are already diagonally similar, then we show that A is either diagonally similar to B or $\mathrm{ct}(B, \overline{S})$ (Lemma 3.7), depending on whether A_1 is diagonally similar to B_1 or B_1^T.

The more interesting case is when A_2 and B_2 are not diagonally similar. Then by induction hypothesis, we assume that A_2 and B_2 are cut-transpose equivalent. In the final step in the proof, we show that we can lift the cut-transpose sequence that relates A_2 and B_2 to a cut-transpose sequence for A and B. This lifting procedure is as follows: for each cut X in the sequence, we either replace it with $X \cup S$ or keep it as it is, depending on whether X contains t or not. We demonstrate this lifting of the cut-transpose sequence via an example. Consider two PME matrices

$$A = \begin{pmatrix} 1 & 3 & 1 & 1 & 1 \\ 2 & 1 & -1 & -1 & -1 \\ 1 & 2 & 2 & 1 & 1 \\ 2 & 4 & -2 & 3 & 4 \\ -1 & -2 & 1 & 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 1 & -2 & 1 \\ 3 & 1 & 2 & -4 & 2 \\ 1 & -1 & 2 & -2 & 1 \\ -1 & 1 & 1 & 3 & 5 \\ -1 & 1 & 1 & 4 & 6 \end{pmatrix}$$

Let us index the rows and columns of these two matrices by $\{a, b, c, d, e\}$. Observe that matrices A and B have common cut $S = \{a, b\}$. We decompose each of them to obtain two smaller matrices as given below. Here matrices A_1 and B_1 are submatrices of A and B, respectively, indexed by $\{a, b, c\}$. Similarly, matrices A_2 and B_2 are submatrices indexed by $\{b, c, d, e\}$.

$$A_1 = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & 2 \end{pmatrix}, \; A_2 = \begin{pmatrix} 1 & -1 & -1 & -1 \\ 2 & 2 & 1 & 1 \\ 4 & -2 & 3 & 4 \\ -2 & 1 & 5 & 6 \end{pmatrix}, \; B_1 = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & 2 \\ 1 & -1 & 2 \end{pmatrix}, \; B_2 = \begin{pmatrix} 1 & 2 & -4 & 2 \\ -1 & 2 & -2 & 1 \\ 1 & 1 & 3 & 5 \\ 1 & 1 & 4 & 6 \end{pmatrix}$$

Observe that $A_1 = B_1^T$. To relate A_2 and B_2, observe that applying cut-transpose on B_2 along cut $\{b, c\}$, gives us

$$\begin{pmatrix} 1 & 2 & 2 & 2 \\ -1 & 2 & 1 & 1 \\ -2 & -2 & 3 & 4 \\ 1 & 1 & 5 & 6 \end{pmatrix}.$$

The obtained matrix diagonally similar to A_2 (they are related by diagonal matrix $D = \mathrm{diag}(-2, 1, 1, 1)$). Hence, the cut-transpose sequence for A_2 and B_2 is simply $(\{b, c\})$. To lift this sequence to A and B we have to take union with $\{a, b\}$ (because $\{b, c\}$ contains b). That is, we obtain the sequence $(\{a, b, c\})$. Finally, since $A_1 = B_1^T$, we need to append this sequence by another cut $\overline{S} = \{c, d, e\}$. Hence, the cut-transpose sequence relating A and B is $(\{a, b, c\}, \{c, d, e\})$. Following equation shows this.

$$A = \begin{pmatrix} 1 & 3 & 1 & 1 & 1 \\ 2 & 1 & -1 & -1 & -1 \\ 1 & 2 & 2 & 1 & 1 \\ 2 & 4 & -2 & 3 & 4 \\ -1 & -2 & 1 & 5 & 6 \end{pmatrix} \xrightarrow[\text{along } \{a,b,c\}]{\text{cut-transpose}} \begin{pmatrix} 1 & 3 & 1 & -2 & 1 \\ 2 & 1 & -1 & 2 & -1 \\ 1 & 2 & 2 & -2 & 1 \\ -1 & -2 & 1 & 3 & 5 \\ -1 & -2 & 1 & 4 & 6 \end{pmatrix}$$

$$\xrightarrow[\text{along } \{c,d,e\}]{\text{cut-transpose}} \begin{pmatrix} 1 & 2 & 1 & -2 & 1 \\ 3 & 1 & 2 & -4 & 2 \\ 1 & -1 & 2 & -2 & 1 \\ -1 & 1 & 1 & 3 & 5 \\ -1 & 1 & 1 & 4 & 6 \end{pmatrix} = B$$

An efficient algorithm. Now we describe some of the ideas involved in our polynomial time algorithm to find a cut-transpose sequence for two irreducible PME matrices. The lemmas mentioned above all have constructive proofs, that is, the following tasks can be done in polynomial time.

- Given a minimal cut S of matrix A, we can check whether it is also a cut of matrix B. If not, then we can find an appropriate cut in B such that applying cut-transpose along it gives us a matrix that has S as a cut.

- Given a cut-transpose sequence for A_2 and B_2 (as defined above), we can find one for A and B.

Two parts which remain unclear are – (i) how to find a minimal cut of a matrix efficiently and (ii) how to compute $(A+Z)^{\text{adj}}$ efficiently for a given matrix A?

To find a cut of a matrix A, we first show that the function $f(X) := \text{rank}(A[X,\overline{X}]) + \text{rank}(A[\overline{X},X])$ is a submodular function (Lemma 3.8). Then observe that if an irreducible matrix A has a cut, then cuts are precisely those sets X which minimize $f(X)$ under the constraints $|X| \geq 2$ and $|\overline{X}| \geq 2$. To find an inclusion-wise minimal cut, we simply find a minimum size cut, using the known algorithms for submodular function minimization under such constraints (Lemma 3.8).

Coming to the second question, recall that instead of matrix A, we consider $(A+Z)^{\text{adj}}$ to ensure that all matrix entries are nonzero. Here Z is a diagonal matrix with distinct algebraically independent elements (or indeterminates). However, it is not clear if we can compute (or even write down) the entries $(A+Z)^{\text{adj}}$ efficiently (succinctly). For efficiency, we need to replace the diagonal entries in Z with elements from the given field (or a large enough algebraic extension). Using ideas from polynomial identity testing, we show that in (deterministic) polynomial time, we can compute an appropriate matrix Z, which ensures that the entries of $(A+Z)^{\text{adj}}$ are all nonzero (Claim 3.9).

Applications to PIT. As discussed earlier, our algorithm to test principal minor equivalence of two matrices A and B can also be viewed as an algorithm to test if the following is a polynomial identity:

$$\det(A+Y) = \det(B+Y),$$

where Y is a diagonal matrix with its diagonal entries being all distinct variables. Theorem 1.3 considers a more general PIT question: whether $\det(A_0 + A_1y_1 + \ldots + A_my_m) = \det(B_0 + B_1y_1 + \ldots + B_my_m)$ for given rank-1 matrices $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_m$ and arbitrary matrices A_0, B_0. We get a deterministic algorithm for this more general PIT question via a reduction to testing principal minor equivalence of two given matrices (Section 5). The reduction uses matroid intersection as a subroutine, which is known to be solvable in deterministic polynomial time.

2 Notation and Preliminaries

We use $[n]$ to denote the set of positive integers $\{1, 2, \ldots, n\}$. For any $X \subseteq [n]$, $\overline{X}$ denotes the complement set X. For two sets S and T, $S \Delta T$ denotes the symmetric difference of S and T. For a set X and an element e, we use $X + e$ to denote the set $X \cup \{e\}$ and $X - e$ to denote the set $X \setminus \{e\}$.

Suppose that $w_1 = (w_{1,1}, w_{1,2}, \ldots, w_{1,k_1})^T, \ldots, w_\ell = (w_{\ell,1}, w_{\ell,2}, \ldots, w_{\ell,k_\ell})^T$ are ℓ vectors over a field $\mathbb{F}$. Then, we use $(w_1 \mid \cdots \mid w_\ell)$ to denote the concatenation of the vectors $w_1, \ldots, w_\ell$ as follows

$$(w_1 \mid \cdots \mid w_\ell) = (w_{1,1}, \ldots, w_{1,k_1}, \ldots w_{\ell,1}, \ldots, w_{\ell,k_\ell})^T.$$

For an $n \times n$ matrix A and $S, T \subseteq [n]$, $A[S,T]$ denotes the submatrix of A with rows indexed by elements in S and columns indexed by elements in T. For $S \subseteq [n]$, let $A[S]$ denote the submatrix $A[S,S]$. When $S = \{i\}$, then $A[i,T] = A[S,T]$. We follow a similar notation when T is a singleton. For a square matrix A, by A^{adj}, we denote the adjoint, or adjugate, of A.

2.1 Principal minor equivalence

Suppose that A and B are two $n \times n$ matrices over any field. The matrix A is said to be *principal minor equivalent* to B if the corresponding principal minors of A and B are *equal*, i.e. for all

$S \subseteq [n]$, $\det(A[S,S]) = \det(B[S,S])$. We use $A \overset{\text{PME}}{=} B$ to denote that A is the principal minor equivalent to B.

The following lemma shows that the principal minor equivalence relation between two matrices remains unchanged under adjoint operation and shift by appropriate diagonal matrices. It is a straightforward consequence of [HL84, Lemma 4].

Lemma 2.1. *Let A and B be two $n \times n$ matrices over a field $\mathbb{F}$. Let D be an $n \times n$ diagonal matrix over $\mathbb{F}$ such that $A + D$ and $B + D$ are non-singular. Then, $A \overset{\text{PME}}{=} B$ if and only if $(A + D)^{\text{adj}} \overset{\text{PME}}{=} (B + D)^{\text{adj}}$.*

2.2 Reducible and Irreducible matrix

Definition 2.2 (Reducible and Irreducible matrix)**.** A matrix is called *reducible* if it can be written as a block upper triangular matrix after permuting the rows and the corresponding columns. A matrix that is not reducible is called *irreducible*.

Equivalently, if we replace the nonzero off-diagonal entries with one and the diagonal entries with zero, then a reducible matrix corresponds to the adjacency matrix of a directed graph having more than one strongly connected component.

From the above definition, it is easy to see that any matrix A with all nonzero off-diagonal entries is an irreducible matrix. The above definition directly gives us the following observation.

Observation 2.3. *Let A be an $n \times n$ matrix over a field $\mathbb{F}$ such that the row and columns of A are indexed by $[n]$. Let G_A be a directed graph defined as follows: the vertex set in $[n]$, and a tuple (i, j) is an edge of G_A if and only if $i \neq j$ and $A[i, j] \neq 0$. Let $I_1, I_2, \ldots, I_s$ be the strongly connected components of A. Then, after permuting the rows and the corresponding columns, the matrix A can be made a block upper triangular matrix, and the diagonal blocks $A(I_1), A(I_2), \ldots, A(I_s)$ are irreducible matrices.*

For two reducible matrices A and B, the next lemma helps to reduce the testing of whether $A \overset{\text{PME}}{=} B$ to multiple instances of testing whether two irreducible matrices have the same corresponding principal minors. The following lemma is a direct consequence of [Ahm23, Corollary 5.4].

Lemma 2.4. *Let A and B two $n \times n$ matrices over a field $\mathbb{F}$. Suppose that after permuting the rows and the corresponding columns, A can be written as a block upper triangular matrix with s diagonal blocks $A_1, A_2, \ldots, A_s$ where each A_i is irreducible and the rows and columns of A_i are indexed by set $T_i \subseteq [n]$. Then, $A \overset{\text{PME}}{=} B$ if and only if the following holds.*

1. *After permuting some rows and the corresponding columns, B can be written as a block upper triangular matrix with s diagonal blocks $B_1, B_2, \ldots, B_s$ where each B_i is irreducible and the rows and columns of B_i are indexed by set T_i.*
2. *For each $i \in [s]$, $A_i \overset{\text{PME}}{=} B_i$.*

2.3 Cut of a matrix

Definition 2.5 (Cut of a matrix)**.** Let A be an $n \times n$ matrix over a field $\mathbb{F}$ such that $n \geq 4$. A subset $X \subset [n]$ is called a *cut* in A if $2 \leq |X| \leq n - 2$ and the rank of the submatrices $A[X, \overline{X}]$ and $A[\overline{X}, X]$ are at most one.

In particular, if A is an irreducible matrix and X is cut in A, then $\text{rank}(A[X, \overline{X}]) = \text{rank}(A[\overline{X}, X]) = 1$.

For an $n \times n$ matrix A, a cut X in A is called a *minimal cut* if there exists no other cut X' in A such that $X' \subseteq X$. Note that any cut of size two is always a minimal cut. Since we can determine the rank of a matrix in polynomial time, we arrive at the following observation.

Observation 2.6. *Given an $n \times n$ matrix A over $\mathbb{F}$ and $X \subseteq [n]$ with $2 \le |X| \le n-2$, it can be decided in polynomial time whether X is a cut in A.*

Next, we show that the set of cuts remains the same after shifting any matrix by an appropriate diagonal matrix.

Lemma 2.7. *Let A be an $n \times n$ matrix over a field $\mathbb{F}$. Let D be an $n \times n$ diagonal matrix over $\mathbb{F}$ such that $A + D$ is non-singular. Then, A and $(A+D)^{\mathrm{adj}}$ have the same set of cuts.*

For proof, see Appendix A.

2.4 Diagonal similarity

Suppose that A and B are two $n \times n$ matrices over a field $\mathbb{F}$. We say that A is *diagonally similar* to B, denoted by $A \overset{\mathrm{DS}}{=} B$, if there exists an $n \times n$ invertible diagonal matrix D over $\mathbb{F}$ such that $B = DAD^{-1}$.

In the following claim, we describe how to efficiently check whether two matrices are diagonally similar or not.

Claim 2.8. *Given two $n \times n$ matrices A and B over $\mathbb{F}$, in polynomial time, we can decide whether $A \overset{\mathrm{DS}}{=} B$.*

Proof Sketch. Observe that if $A \overset{\mathrm{DS}}{=} B$ then we must have an invertible diagonal matrix D such that $B[i,j]/A[i,j] = D[i]/D[j]$, for any $i \neq j$ with $A[i,j] \neq 0$. Consider a weighted directed graph G on n vertices such that (i,j) is an edge for $i \neq j$ if and only if $A[i,j] \neq 0$ or $A[j,i] \neq 0$. Let us define the weight of an edge (i,j) as $w(i,j) = B[j,i]/A[j,i]$ or $A[i,j]/B[i,j]$ whichever is defined. If both are defined, they must be equal, otherwise A and B cannot be diagonally similar. Observe that for any path $(i_0, i_1, \ldots, i_k)$ in graph G, it must be that

$$D[i_k]/D[i_0] = w(i_0,i_1)w(i_1,i_2)\cdots w(i_{k-1},i_k)$$

Moreover, any diagonal matrix satisfying the above equation for all paths in G will give us the desired diagonal matrix D. So, we construct D for given matrices A and B as follows. For any connected component in G, pick an arbitrary vertex i from the component and set $D[i] = 1$. For any other vertex j in that component, find a path $(i = i_0, i_1, i_2, \ldots, i_k = j)$ and set $D[j] = w(i_0,i_1)w(i_1,i_2)\cdots w(i_{k-1},i_k)$. Repeating this for every component in G will give us matrix D. Finally we should check that $B[i,j]/A[i,j] = D[i]/D[j]$, for every $i \neq j$ with $A[i,j] \neq 0$.

□

One can observe that if $A \overset{\mathrm{DS}}{=} B$ or $A \overset{\mathrm{DS}}{=} B^T$, then $A \overset{\mathrm{PME}}{=} B$. Next, we consider the converse direction. Hartfiel and Loewy [HL84, Theorem 3] showed that when $n = 2$ or 3, and A is an irreducible matrix, $A \overset{\mathrm{PME}}{=} B$ implies that $A \overset{\mathrm{DS}}{=} B$ or $A \overset{\mathrm{DS}}{=} B^T$. Later, Lowey [Loe86, Theorem 1] showed that if A is an irreducible matrix and has no cut, then $A \overset{\mathrm{PME}}{=} B$ implies $A \overset{\mathrm{DS}}{=} B$. Therefore, by combining them, we have the following lemma:

Lemma 2.9. *Let A and B two $n \times n$ matrices over a field $\mathbb{F}$ such that A is irreducible and $A \overset{\mathrm{PME}}{=} B$. Then, the following holds:*

1. *if $n = 2$ or 3, then $A \overset{\mathrm{DS}}{=} B$ or $A \overset{\mathrm{DS}}{=} B^T$.*
2. *if $n \geq 4$ and A has no cut, then $A \overset{\mathrm{DS}}{=} B$ or $A \overset{\mathrm{DS}}{=} B^T$.*

The next lemma shows that the diagonal similarity relation carries over through the adjoint. It directly follows from [HL84, Lemma 4].

Lemma 2.10. *Let A and B be two $n \times n$ matrices over $\mathbb{F}$. Let D be an $n \times n$ diagonal matrix such that both $A + D$ and $B + D$ are invertible. Then, $A \overset{\mathrm{DS}}{=} B$ if and only if $(A+D)^{\mathrm{adj}} \overset{\mathrm{DS}}{=} (B+D)^{\mathrm{adj}}$.*

2.5 Cut-transpose operation

In the previous section, we have seen that under diagonal similarity, the values of the principal minors of a matrix remain unchanged. Now, we describe another operation under which also the values of the principal minors remain the same. This operation was defined by Ahmadieh [Ahm23, Lemma 4.5], and we refer to it as cut-transpose.

Definition 2.11 (Cut-transpose operation)**.** Let A be an $n \times n$ matrix, and $X \subseteq [n]$ such that $1 \leq |X| \leq n-1$ and $\operatorname{rank}(A[X, \overline{X}]) \leq 1$ and $\operatorname{rank}(A[\overline{X}, X]) \leq 1$. Let the matrix A be written as

$$\begin{pmatrix} A(X) & p \cdot q^T \\ u \cdot v^T & A(\overline{X}) \end{pmatrix},$$

where $p, v \in \mathbb{F}^{|X|}$ and $q, u \in \mathbb{F}^{|\overline{X}|}$ Then, the *cut-transpose operation on A with respect to X* transforms A to a new matrix $\widetilde{A}$ as follows:

$$\widetilde{A} = \begin{pmatrix} A(X) & p \cdot u^T \\ q \cdot v^T & A(\overline{X})^T \end{pmatrix}.$$

Remark 2.12. Note that in the above definition, for every nonzero $\alpha, \beta \in \mathbb{F}$, the rank-one submatrices $A[X, \overline{X}]$ and $A[\overline{X}, X]$ are equal to $(\alpha p) \cdot (q/\alpha)^T$ and $(\beta u) \cdot (v/\beta)^T$, respectively. Depending on what rank one decomposition we choose, we can get a different matrix after applying the cut-transpose operation, and let $\operatorname{ct}(A, X)$ be exactly the set of all such matrices. However, all the matrices in $\operatorname{ct}(A, X)$ are *diagonally similar* to each other. Therefore, all the matrices in $\operatorname{ct}(A, X)$ have the same corresponding principal minors. Thus, slightly abusing the notation, we also use $\operatorname{ct}(A, X)$ to denote any matrix we can get after applying the cut-transpose operation on A with respect to X. In particular, if $|X| = n-1$ then $\operatorname{ct}(A, X) = A$, and if $|X| = 1$ then $\operatorname{ct}(A, X) = A^T$.

For a $k \times \ell$ matrix M with $\operatorname{rank}(M) \leq 1$, in polynomial time, we can find $p \in \mathbb{F}^k$ and $q \in \mathbb{F}^\ell$ such $M = p \cdot q^T$. Thus, we have the following observation.

Observation 2.13. *Let A be an $n \times n$ matrix over F, and $X \subseteq [n]$ such that* $\operatorname{rank}(A[X, \overline{X}]) \leq 1$ *and* $\operatorname{rank}(A[\overline{X}, X]) \leq 1$. *Then, given A and X, we can compute* $\operatorname{ct}(A, X)$ *in polynomial time.*

Now, we mention some properties of the cut-transpose operation. First, we show that under cut-transpose operation, the values of the principal minors of a matrix remain the same.

Lemma 2.14. *Let A be an $n \times n$ matrix over a field $\mathbb{F}$. Let $X \subseteq [n]$ be a cut in A. Then, $A \overset{\text{PME}}{=} \operatorname{ct}(A, X)$.*

For proof, see Appendix A. Next, we show that the cut-transpose operation on a matrix carries over to the adjoint of A.

Lemma 2.15. *Let A be an $n \times n$ matrix over a field $\mathbb{F}$. Then, for any $X \subseteq [n]$ with* $\operatorname{rank}(A[X, \overline{X}]) \leq 1$ *and* $\operatorname{rank}(A[\overline{X}, X]) \leq 1$,

$$\operatorname{ct}(A, X)^{\text{adj}} = \operatorname{ct}(A^{\text{adj}}, X).$$

From Remark 2.12, $\operatorname{ct}(A, X)$ is a family of matrices such that all the matrices are diagonally similar. The set $\operatorname{ct}(A, X)^{\text{adj}}$ consists of all the matrices we get after taking the adjoint of the matrices in $\operatorname{ct}(A, X)$. For nonzero $\alpha, \beta \in \mathbb{F}$, let $D_{\alpha,\beta}$ be an $n \times n$ diagonal matrix defined as follows: $D_{\alpha,\beta}[i, i] = \alpha$ for $i \in X$ and β otherwise. Then, from Remark 2.12 and [HL84, Lemma 4], if $\widetilde{A} \in \operatorname{ct}(A, X)$,

$$\operatorname{ct}(A, X)^{\text{adj}} = \{D_{\alpha,\beta} \cdot \widetilde{A}^{\text{adj}} \cdot D_{\alpha,\beta}^{-1} \mid \alpha, \beta \in \mathbb{F} \setminus \{0\}\}.$$

On the other hand, from Claim A.1, both $\text{rank}(A^{\text{adj}}[X, \overline{X}])$ and $\text{rank}(A^{\text{adj}}[\overline{X}, X])$ are at most one. Thus, if $A' \in \text{ct}(A^{\text{adj}}, X)$,

$$\text{ct}(A^{\text{adj}}, X) = \{D_{\alpha,\beta} \cdot A' \cdot D_{\alpha,\beta}^{-1} \mid \alpha, \beta \in \mathbb{F} \setminus \{0\}\}.$$

In the proof, we show that there exists a common matrix in $\text{ct}(A, X)^{\text{adj}}$ and $\text{ct}(A^{\text{adj}}, X)$, thus implying the above lemma. For proof, see Appendix A. Next, we mention a definition that will be useful in this article.

Definition 2.16. Let A and B be two matrices over a field $\mathbb{F}$, and their rows and columns are indexed by I. Let $\mathcal{X} = (X_1, X_2, \ldots, X_k)$ be a sequence of subsets of I. We say A, B and $\mathcal{X}$ *satisfy property* $\mathcal{P}$ if $\mathcal{X}$ produces a sequence of matrices $(A_0 = A, A_1, A_2, \ldots, A_k)$ with the following property:

$$\forall i \in [k],\ A_i = \text{ct}(A_{i-1}, X_i) \text{ where } X_i \text{ is } \emptyset, I, \text{ or a cut in } A_{i-1}, \text{ and } A_k \overset{\text{DS}}{=} B.$$

3 Proof of Theorem 1.1: Characterizing and Testing of Prinicipal Minor Equivalence for Irreducible Matrices

3.1 Some useful results on cut and cut-transpose operation

Theorem 3.1. *Let A be an $n \times n$ matrix over $\mathbb{F}$ with nonzero off-diagonal entries. Let $S \subseteq [n]$ be a cut in A. Then, for any $T \subseteq [n]$ the following holds.*

1. *If $T \subseteq S$ or $T \subseteq \overline{S}$, then T is a cut in A if and only if T is cut in* $\text{ct}(A, S)$
2. *Otherwise, T is a cut in A if and only if $T \Delta S$ is a cut in* $\text{ct}(A, S)$.

Proof. We start with the proof of the first part of the theorem.

Proof of the first part. Assume that $T \subseteq S$ and T is a cut in A. Then, the matrix A has the following structure:

$$A = \begin{array}{c} \\ T \\ S \setminus T \\ \overline{S} \end{array} \begin{array}{c} \begin{array}{ccc} T & S \setminus T & \overline{S} \end{array} \\ \begin{pmatrix} * & u_1 \cdot v_1^T & u_1 \cdot v_2^T \\ p_1 \cdot q_1^T & * & u_2 \cdot v_2^T \\ p_2 \cdot q_1^T & p_2 \cdot q_2^T & * \end{pmatrix} \end{array}$$

such that

$$u_1, q_1 \in \mathbb{F}^{|T|}, \quad v_1, u_2, p_1, q_2 \in \mathbb{F}^{|S|-|T|}, \quad \text{and} \quad v_2, p_2 \in \mathbb{F}^{|\overline{S}|},$$

and '$*$' marked submatrices can be arbitrary. After applying the cut-transpose operation on A with respect to the cut S,

$$\text{ct}(A, S)[T, \overline{T}] = u_1 \cdot (v_1 \mid p_2)^T \text{ and } \text{ct}(A, S)[\overline{T}, T] = (p_1 \mid v_2) \cdot q_1^T.$$

Therefore, T is also a cut in $\text{ct}(A, S)$.

For the converse direction, observe that S is a cut in $\text{ct}(A, S)$, and A can be seen as a matrix we get after applying the cut-transpose operation on $\text{ct}(A, S)$ with respect to S. Therefore, the above analysis also says that T will be a cut in A if it is a cut in $\text{ct}(A, S)$ and $T \subseteq S$.

Now we assume that $T \subseteq \overline{S}$. Note that the set of cuts in A is the same as the set of cuts in A^T. Since $T \subseteq \overline{S}$, from the above discussion, T is a cut in A^T if and only if T is a cut in $\text{ct}(A^T, \overline{S})$. Observe that $\text{ct}(A^T, \overline{S}) = \text{ct}(A, S)$. Thus, when $T \subseteq \overline{S}$, the set T is a cut in A if and only if T is a cut in $\text{ct}(A, S)$.

Proof of the second part. Assume that T is neither a subset of S nor a subset of $\overline{S}$, and T is a cut in A. This implies that $S \setminus T$, $S \cap T$ and $T \setminus S$ are nonempty. Since S is a cut in A, the matrix A has the following structure.

$$A = \begin{array}{c} \\ S \setminus T \\ S \cap T \\ T \setminus S \\ \overline{S \cup T} \end{array} \begin{array}{c} \begin{array}{cccc} S \setminus T & S \cap T & T \setminus S & \overline{S \cup T} \end{array} \\ \begin{pmatrix} * & * & u_1 \cdot v_1^T & u_1 \cdot v_2^T \\ * & * & u_2 \cdot v_1^T & u_2 \cdot v_2^T \\ p_1 \cdot q_1^T & p_1 \cdot q_2^T & * & * \\ p_2 \cdot q_1^T & p_2 \cdot q_2^T & * & * \end{pmatrix} \end{array} \tag{3}$$

such that

$$u_1, q_1 \in \mathbb{F}^{|S \setminus T|}, \quad v_1, p_1 \in \mathbb{F}^{|T \setminus S|}, \quad v_2, p_2 \in \mathbb{F}^{|\overline{S \cup T}|}, \text{ and } u_2, q_2 \in \mathbb{F}^{|S \cap T|}.$$

On the other hand, the cut T implies the following structural constraint for A.

$$A = \begin{array}{c} \\ S \setminus T \\ S \cap T \\ T \setminus S \\ \overline{S \cup T} \end{array} \begin{array}{c} \begin{array}{cccc} S \setminus T & S \cap T & T \setminus S & \overline{S \cup T} \end{array} \\ \begin{pmatrix} * & c_1 \cdot d_1^T & c_1 \cdot d_2^T & * \\ a_1 \cdot b_1^T & * & * & a_1 \cdot b_2^T \\ a_2 \cdot b_1^T & * & * & a_2 \cdot b_2^T \\ * & c_2 \cdot d_1^T & c_2 \cdot d_2^T & * \end{pmatrix} \end{array} \tag{4}$$

such that

$$b_1, c_1 \in \mathbb{F}^{|S \setminus T|}, \quad a_2, d_2 \in \mathbb{F}^{|T \setminus S|}, \quad b_2, c_2 \in \mathbb{F}^{|\overline{S \cup T}|}, \text{ and } a_1, d_1 \in \mathbb{F}^{|S \cap T|}.$$

Since the off entries of A are nonzero, comparing rank-one submatrices in Eq. (3) and Eq. (4), the following holds:

1. From $A[S \setminus T, T \setminus S]$, $b_1 = \alpha \cdot q_1$ and $a_2 = \alpha^{-1} \cdot p_1$ some nonzero $\alpha \in \mathbb{F}$.
2. From $A[\overline{S \cup T}, S \cap T]$, $d_1 = \beta \cdot q_2$ and $c_2 = \beta^{-1} \cdot p_2$ for some nonzero $\beta \in \mathbb{F}$.
3. From $A[T \setminus S, S \setminus T]$, $d_2 = \gamma \cdot v_1$ and $c_1 = \gamma^{-1} \cdot u_1$ for some nonzero $\gamma \in \mathbb{F}$.
4. From $A[S \cap T, \overline{S \cup T}]$, $b_2 = \delta \cdot v_2$ and $a_1 = \delta^{-1} \cdot u_2$ for some nonzero $\beta \in \mathbb{F}$.

Thus, using Eq. (3) and Eq. (4), the matrix A has the following form.

$$A = \begin{array}{c} \\ S \setminus T \\ S \cap T \\ T \setminus S \\ \overline{S \cup T} \end{array} \begin{array}{c} \begin{array}{cccc} S \setminus T & S \cap T & T \setminus S & \overline{S \cup T} \end{array} \\ \begin{pmatrix} * & \zeta(u_1 \cdot q_2^T) & u_1 \cdot v_1^T & u_1 \cdot v_2^T \\ \omega^{-1}(u_2 \cdot q_1^T) & * & u_2 \cdot v_1^T & u_2 \cdot v_2^T \\ p_1 \cdot q_1^T & p_1 \cdot q_2^T & * & \omega(p_1 \cdot v_2^T) \\ p_2 \cdot q_1^T & p_2 \cdot q_2^T & \zeta^{-1}(p_2 \cdot v_1^T) & * \end{pmatrix} \end{array} \tag{5}$$

where $\zeta = \beta\gamma^{-1}$ and $\omega = \delta\alpha^{-1}$. From Eq. (5), applying cut-transpose operation on A with respect to the cut S, we get that

$$\begin{aligned} \mathrm{ct}(A,S)[T\Delta S, \overline{T\Delta S}] &= (\zeta u_1 \mid v_1) \cdot (q_2 \mid \zeta^{-1} p_2)^T, \text{ and} \\ \mathrm{ct}(A,S)[\overline{T\Delta S}, T\Delta S] &= (\omega^{-1} u_2 \mid v_2) \cdot (q_1 \mid \omega p_1)^T. \end{aligned}$$

Therefore, $S\Delta T$ is a cut in $\mathrm{ct}(A,S)$.

For the converse direction, assume that $T\Delta S$ is a cut in $\mathrm{ct}(A,S)$. As mentioned earlier, A can be see as the matrix we get after applying cut-transpose operation on $\mathrm{ct}(A,S)$ with respect to the cut S. Therefore, the above discussion implies that if $T\Delta S$ is a cut in $\mathrm{ct}(A,S)$, then $(T\Delta S)\Delta S = T$ is a cut in A. Thus, when T is neither a subset of S nor a subset of $\overline{S}$, T is a cut in A if and only in $T\Delta S$ is a cut in $\mathrm{ct}(A,S)$. This completes the proof of our theorem. □

Lemma 3.2. *Let A be a 4×4 matrix over $\mathbb{F}$ with all off-diagonal entries are nonzero. Let B be another 4×4 matrix over $\mathbb{F}$ such that $A \overset{\text{PME}}{=} B$. Then, one of the following two holds:*

1. *$A \overset{DS}{=} B$ or $A \overset{DS}{=} B^T$.*
2. *The exists a common cut in A and B. Furthermore, for any common cut X of A and B, $\mathrm{ct}(A,X) \overset{DS}{=} B$ or $\mathrm{ct}(A,X) \overset{DS}{=} B^T$.*

Lemma 3.3. *Let A and B be two $n\times n$ matrices over field $\mathbb{F}$ with nonzero off-diagonal entries such that $A \overset{\text{PME}}{=} B$ and $S = \{s_1, s_2\}$ is a cut in A of size 2. Then either S is a cut in B or X_i for each $i \in \{1,2\}$, defined as follows*

$$X_i = \{t \in \overline{S} \mid A(S+t) \overset{DS}{=} B(S+t)\} \cup \{s_i\},$$

is a cut in B such that S is a cut in $\mathrm{ct}(B, X_i)$.

Proof. Without loss of generality, let $S = \{1,2\}$. Let $3 \le t \le n$. Since $A \overset{\text{PME}}{=} B$, it follows that $A(\{1,2,t\}) \overset{\text{PME}}{=} B(\{1,2,t\})$. From Lemma 2.9, we have $A(\{1,2,t\}) \overset{\text{DS}}{=} B(\{1,2,t\})$ or $B(\{1,2,t\})^T$. Hence, there exists a diagonal matrix D_t with $D_t[1,1] = 1$ such that

$$D_t A(\{1,2,t\}) D_t^{-1} = B(\{1,2,t\}) \text{ or}$$

$$D_t A(\{1,2,t\}) D_t^{-1} = B(\{1,2,t\})^T.$$

For any t for which the former condition holds, we will have

$$\frac{B[1,t]}{B[2,t]} = \frac{A[1,t]A[2,1]}{A[2,t]B[2,1]} = \frac{A[1,3]A[2,1]}{A[2,3]B[2,1]} \text{ and} \tag{6}$$

$$\frac{B[t,1]}{B[t,2]} = \frac{A[t,1]A[1,2]}{A[t,2]B[1,2]} = \frac{A[3,1]A[1,2]}{A[3,2]B[1,2]}. \tag{7}$$

The last two equalities hold because $\{1,2\}$ is a cut in A. For any t for which the later condition holds, we will have

$$\frac{B[1,t]}{B[2,t]} = \frac{A[t,1]A[1,2]}{A[t,2]B[2,1]} = \frac{A[3,1]A[1,2]}{A[3,2]B[2,1]} \text{ and} \tag{8}$$

$$\frac{B[t,1]}{B[t,2]} = \frac{A[1,t]A[2,1]}{A[2,t]B[1,2]} = \frac{A[1,3]A[2,1]}{A[2,3]B[1,2]}. \tag{9}$$

If equations (6) and (7) hold for every $3 \le t \le n$, or if equations (8) and (9) hold for every $3 \le t \le n$, then $\{1,2\}$ will be a cut of B.

Suppose that is not true. It follows that

$$\frac{A[1,3]A[2,1]}{A[2,3]} \ne \frac{A[3,1]A[1,2]}{A[3,2]}.$$

Let $P \subseteq \{3,4,\ldots,n\}$ be the set of indices for which equations (6), (7) hold and let $Q := \{3,4,\ldots,n\} \setminus P$ be the set of indices for which equations (8), (9) hold.

We will show that $P \cup \{1\}$ is a cut in B. Consider two indices $s \in P$ and $t \in Q$. Consider the set $T = \{1,2,s,t\}$. Since equations (6) and (7) hold for s and do not hold for t, we have that

$$\frac{B[1,t]}{B[2,t]} \neq \frac{B[1,s]}{B[2,s]} \text{ or } \frac{B[t,1]}{B[t,2]} \neq \frac{B[s,1]}{B[s,2]}.$$

Hence, $\{1,2\}$ is not a cut in $B[T]$ and $B[T] \overset{\text{DS}}{\neq} A[T]$. But, we have that $A[T] \overset{\text{PME}}{=} B[T]$. Hence, there must be a cut in $B[T]$ (Lemma 3.2) In fact, $B[T]$ will have more than one cut. Because if $B[T]$ has a unique cut, say $\{1,t\}$, then that will also be a unique cut of $A[T]$ (Lemma 3.2). But, $A[T]$ has a cut $\{1,2\}$.

So, we conclude that $B[T]$ has cuts $\{1,s\}$ and $\{1,t\}$. Hence, we can write

$$\begin{aligned} B[s,t]/B[1,t] &= B[s,2]/B[1,2] \text{ and} \\ B[t,s]/B[2,s] &= B[t,1]/B[2,1] \end{aligned}$$

Using these equations for every $s \in P$ and every $t \in Q$, we get that $X = P \cup \{1\}$ is a cut in B. Similarly, we can show that $X' = P \cup \{2\}$ is a cut in B. From Theorem 3.1, $X \Delta X' = \{1,2\}$ is a cut of $\mathrm{ct}(B,X)$ and $\mathrm{ct}(B,X')$. □

Lemma 3.4. *Let A be an $n \times n$ matrix over $\mathbb{F}$ such that the off-diagonal entries of A are nonzero. Let S be a minimal cut in A of size greater than two. Let T be a nonempty subset of $\overline{S}$, $X \subseteq S \cup T$ and $\widetilde{X} = (S \cup T) \setminus X$. Then, if X is a cut in $A(S \cup T)$, then either $S \subseteq X$ or $S \subseteq \widetilde{X}$.*

In particular, if $T = \{t\}$ for some $t \in \overline{S}$, then the matrix $A(S+t)$ have no cut.

Proof. For the sake of contradiction, assume that $X \subseteq S \cup T$ is a cut of $A(S+T)$ such that neither $S \subseteq X$ nor $S \subseteq \widetilde{X}$. Since $|S| \geq 3$, either $|S \cap X| \geq 2$ or $|S \setminus X| \geq 2$. Next, we divide our proof into the following four cases.

Case I: We assume that $|S \cap X| \geq 2$ and $|T \setminus X| \neq \emptyset$. Then, we show that $S \cap X$ is cut in A. This will violate the minimality of S.

Case II: We assume that $|S \cap X| \geq 2$ and $|T \setminus X| = \emptyset$. This implies that $T \subseteq X$, hence $\widetilde{X} \subseteq S$. Therefore, $|S \setminus X| \geq 2$ and $T \cap X$ is nonempty. In this case, we show that $S \setminus X$ is cut in A, which will again violate the minimality of S.

Case III: We assume that $|S \setminus X| \geq 2$ and $T \cap X \neq \emptyset$. In this case, we show that $S \setminus X$ is a cut in A which will violate the minimality of S.

Case IV: We assume that $|S \setminus X| \geq 2$ and $T \cap X = \emptyset$. Like **Case II**, this implies that $|S \cap X| \geq 2$ and $T \setminus X$ is nonempty. In this case, we show that $S \cap X$ is a cut in A. This will violate the minimality of S.

The proof of all the above four cases is similar. We divide the index set $S \cup T$ of the matrix $A(S \cup T)$ into the four following disjoint subsets

$$S \cap X, S \setminus X, T \cap X, T \setminus X,$$

and divide the index set $[n]$ of the matrix A into the five following disjoint subsets

$$S \cap X, S \setminus X, T \cap X, T \setminus X, \overline{S \cup T}.$$

The cut S will put some structural constraint on A and the cut X will put some structural constraints on $A(S \cup T)$. As we did in the proof of Theorem 3.1, by comparing the same rank one submatrices, we show an appropriate subset is a cut in A. Here, we only mention the detail proof of **Case I**. The other cases can be proven similarly.

Case I. The cut X in $A(S \cup T)$ gives the following structural constraint.

$$A(S \cup T) = \begin{array}{c} \\ S \cap X \\ S \setminus X \\ T \cap X \\ T \setminus X \end{array} \begin{array}{c} \begin{array}{cccc} S \cap X & S \setminus X & T \cap X & T \setminus X \end{array} \\ \begin{pmatrix} * & a_1 \cdot b_1^T & * & a_1 \cdot b_2^T \\ c_1 \cdot d_1^T & * & c_1 \cdot d_2^T & * \\ * & a_2 \cdot b_1^T & * & a_2 \cdot b_2^T \\ c_2 \cdot d_1^T & * & c_2 \cdot d_2^T & * \end{pmatrix} \end{array} \tag{10}$$

where

$$a_1, d_1 \in \mathbb{F}^{|S \cap X|}, \;\; b_1, c_1 \in \mathbb{F}^{|S \setminus X|}, \;\; a_2, d_2 \in \mathbb{F}^{|T \cap X|}, \;\; \text{and} \;\; b_2, c_2 \in \mathbb{F}^{|T \setminus X|},$$

and '$*$' marked submatrices can be arbitrary. As S is a cut in A, we have the following structure of A.

$$A = \begin{array}{c} \\ S \cap X \\ S \setminus X \\ T \cap X \\ T \setminus X \\ \overline{S \cup T} \end{array} \begin{array}{c} \begin{array}{ccccc} S \cap X & S \setminus X & T \cap X & T \setminus X & \overline{S \cup T} \end{array} \\ \begin{pmatrix} * & * & p_1 \cdot q_1^T & p_1 \cdot q_2^T & p_1 \cdot q_3^T \\ * & * & p_2 \cdot q_1^T & p_2 \cdot q_2^T & p_2 \cdot q_3^T \\ u_1 \cdot v_1^T & u_1 \cdot v_2^T & * & * & * \\ u_2 \cdot v_1^T & u_2 \cdot v_2^T & * & * & * \\ u_3 \cdot v_1^T & u_3 \cdot v_2^T & * & * & * \end{pmatrix} \end{array} \tag{11}$$

where

$$p_1, v_1 \in \mathbb{F}^{|S \cap X|}, \;\; p_2, v_2 \in \mathbb{F}^{|S \setminus X|}, \;\; q_1, u_1 \in \mathbb{F}^{|T \cap X|}, \;\; q_2, u_2 \in \mathbb{F}^{|T \setminus X|}, \;\; \text{and} \;\; u_3, q_3 \in \mathbb{F}^{|\overline{S \cup T}|}.$$

Note that $T \setminus X$ is nonempty. Since the off-digonal entries of A are nonzero, comparing the rank-one submatrices in Eq. (10) and Eq. (11), we get

1. From $A[T \setminus X, S \cap X]$, $d_1 = \alpha v_1$ for some nonzero $\alpha \in \mathbb{F}$.
2. From $A[S \cap X, T \setminus X]$, $a_1 = \beta p_1$ for some nonzero $\beta \in \mathbb{F}$.

This combined with Eq. (10) and Eq. (11), we have that

$$\begin{aligned} A[S \cap X, \overline{S \cap X}] &= p_1 \cdot (\beta b_1 \mid q_1 \mid q_2 \mid q_3)^T, \text{ and} \\ A[\overline{S \cap X}, S \cap X] &= (\alpha c_1 \mid u_1 \mid u_2 \mid u_3) \cdot v_1^T. \end{aligned}$$

Thus, $S \cap X$ is cut in A which violates the minimality of S. This completes the proof of **Case I**. Like **Case I**, we can prove the other three cases. The details are omitted here.

Now we prove the other part of the lemma. Suppose this T is a singleton set, i.e. $T = \{t\}$ for some $t \in \overline{S}$. For the sake of contradiction, assume that there exists a cut X in $A(S + t)$. Then, from the first part of the lemma, either $S \subseteq X$ or $S \subseteq \widetilde{X}$ where $\widetilde{X} = (S + t) \setminus X$. Without loss of generality, assume $S \subseteq X$. Then $|\widetilde{X}| \leq 1$. This is a contradiction since X is a cut in $A(S + t)$. Therefore, $A(S \cup T)$ has no cut when T is a singleton set. □

Lemma 3.5. *Let A be an $n \times n$ matrix over $\mathbb{F}$ with nonzero off-diagonal entries. Let $S \subseteq [n]$ be a cut in the matrix A and $t \in S$, and $X \subseteq \overline{S}$ is a cut in $A(\overline{S} + t)$. Then, X is also a cut in the matrix A.*

Proof. The off-diagonal entries of A are nonzero. The sets X and S are cuts in $A(S+t)$ and A, respectively. This implies that the matrix A can be written as follows.

$$A = \begin{array}{c} \\ X \\ \overline{S}\setminus X \\ t \\ S-t \end{array} \begin{array}{c} \begin{array}{cccc} X & \overline{S}\setminus X & t & S-t \end{array} \\ \begin{pmatrix} * & u_1 \cdot v_1^T & u_1 & u_1 \cdot v_2^T \\ p_1 \cdot q_1^T & * & u_2 & u_2 \cdot v_2^T \\ q_1^T & q_2^T & * & * \\ p_2 \cdot q_1^T & p_2 \cdot q_2^T & * & * \end{pmatrix} \end{array}$$

where

$$u_1, q_1 \in \mathbb{F}^{|X|}, \quad v_1, u_2, p_1, q_2 \in \mathbb{F}^{|\overline{S}\setminus X|}, \quad v_2, p_2 \in \mathbb{F}^{|S|-1},$$

and '$*$'marked submatrices can be arbitrary. From the above structure of A, observe that

$$A[X, \overline{X}] = u_1 \cdot (v_1 \mid 1 \mid v_2)^T \text{ and } A[\overline{X}, X] = (p_1 \mid 1 \mid p_2) \cdot q_1^T.$$

Therefore, X is a cut in A. □

Lemma 3.6. *Let A and B be two $n \times n$ matrices over $\mathbb{F}$ with off-diagonal entries are nonzero. Let $A \overset{\text{PME}}{=} B$, and $S \subseteq n$ be a minimal cut in A of size greater than two. Then, S is also a cut in B.*

Proof. Let $s \in \overline{S}$. We show that for all $t \in \overline{S+s}$, the set $T_t := \{s, t\}$ is a cut in $B(S + T_t)$. This will imply that

$$B[S, t] = \alpha \cdot B[S, s] \text{ and } B[t, S] = \beta \cdot B[s, S]$$

for some $\alpha, \beta \in \mathbb{F}$. Hence, S is a cut in B.

Since S is a minimal cut in A of size greater than two, from Lemma 3.5, there are no cuts in both the matrices $A(S+s)$ and $A(S+t)$. We have that $A \overset{\text{PME}}{=} B$. Therefore, applying Lemma 2.9, $A(S+s) \overset{\text{DS}}{=} B(S+s)$ and $A(S+t) \overset{\text{DS}}{=} B(S+t)$. This implies that both $B(S+s)$ and $B(S+t)$ have no cuts.

For the sake of contradiction, assume that T_t is not a cut in $B(S+T_t)$. Note that T_t is a cut in $A(S+T_t)$ of size two. Then, from Lemma 3.3, there exists a cut $X \subseteq S + T_t$ in the matrix $B(S+T_t)$ such that $s \in X$ but $t \notin X$. Since $|S + T_t| \geq 5$, either $|X| > 2$ or the size of $\widetilde{X} := (S + T_t) \setminus X$ is greater than 2. If $|X| > 2$, then $X - s$ is a cut in $B(S+t)$. Otherwise, $\widetilde{X} - t$ is a cut in $B(S+s)$. In both the cases, we have contradictions. Thus, T_t is a cut in $B(S+T_t)$ for all $t \in \overline{S+s}$. This completes our proof. □

Lemma 3.7. *Let A and B be two $n \times n$ matrices over $\mathbb{F}$ with nonzero off-diagonal entries and $A \overset{\text{PME}}{=} B$. Let $S \subseteq [n]$ be a minimal cut in A and also a cut in B. Let $s \in S$ such that $A(\overline{S}+s) \overset{\text{DS}}{=} B(\overline{S}+s)$. Then, either $A \overset{\text{DS}}{=} B$ or $\mathrm{ct}(A, \overline{S}) \overset{\text{DS}}{=} B$.*

Proof. Without loss of generality, assume that $S = [i]$ and $s = i$. Then, from the hypothesis, $B(\overline{S}+i) \overset{\text{PME}}{=} A(\overline{S}+i)$. Since S is a minimal cut in A, using Lemma 3.4 and Lemma 2.9, there exists an $(i+1) \times (i+1)$ invertible diagonal matrix D_1 such that $D_1[i+1, i+1] = 1$ and

$$D_1 \cdot A([i+1]) \cdot D_1^{-1} = B([i+1]) \text{ or } B([i+1])^T.$$

From the hypothesis, there exists another $(n-i+1) \times (n-i+1)$ invertible diagonal matrix D_2 such that $D_2[i, i] = 1$ and

$$B(\overline{S}+i) = D_2 \cdot A(\overline{S}+i) \cdot D_2^{-1}. \tag{12}$$

We assume that the rows and columns of D_2 are indexed by $\overline{S}+i$. Next, we divide our proof into the following two cases.

Case I: In this case, we assume that

$$D_1 \cdot A([i+1]) \cdot D_1^{-1} = B([i+1]), \tag{13}$$

and show $A \overset{\text{DS}}{=} B$. Let D be an $n \times n$ invertible matrix defined as follows: For all $k \in [n]$,

$$D[k,k] = \begin{cases} D_1[k,k] & \text{if } k \in [i] \\ \frac{D_2[k,k]}{D_2[i+1,i+1]} & \text{otherwise .} \end{cases}$$

We will show that B is equal to DAD^{-1}. Since S is a common cut in both the matrices A and B, the rank-one submatrices $A[S,\overline{S}]$ and $B[S,\overline{S}]$ can be written as follows.

$$A[S,\overline{S}] = A[S,i+1] \cdot \frac{A[i,\overline{S}]}{A[i,i+1]} \text{ and } A[\overline{S},S] = A[\overline{S},i] \cdot \frac{A[i+1,S]}{A[i+1,i]} \tag{14}$$

$$B[S,\overline{S}] = B[S,i+1] \cdot \frac{B[i,\overline{S}]}{B[i,i+1]} \text{ and } B[\overline{S},S] = B[\overline{S},i] \cdot \frac{B[i+1,S]}{B[i+1,i]} \tag{15}$$

From Eq. (12) and Eq. (13),

$$\begin{aligned} B[i,i+1] &= A[i,i+1] \cdot D_2^{-1}[i+1,i+1] \\ B[i,\overline{S}] &= A[i,\overline{S}] \cdot D_2^{-1}(\overline{S}), \text{ and} \\ B[S,i+1] &= D_1(S) \cdot A[S,i+1] \end{aligned}$$

Therefore, using the above equation and Eq. (15),

$$\begin{aligned} B[S,\overline{S}] &= D_1(S) \cdot A[S,i+1] \cdot \frac{D_2[i+1,i+1] \cdot A[i,\overline{S}] \cdot D_2^{-1}(\overline{S})}{A[i,i+1]} \\ &= D(S) \cdot A[S,\overline{S}] \cdot D^{-1}(\overline{S}) \end{aligned}$$

Similarly, we can show that

$$B[\overline{S},S] = D(\overline{S}) \cdot A[\overline{S},S] \cdot D^{-1}(S).$$

Applying Eq. (13) and Eq. (12), we get that

$$\begin{aligned} B(S) &= D(S) \cdot A(S) \cdot D^{-1}(S) \text{ and} \\ B(\overline{S}) &= D(\overline{S}) \cdot A(\overline{S}) \cdot D^{-1}(\overline{S}). \end{aligned}$$

Thus, $B = DAD^{-1}$.

Case II: In this case, we assume that

$$D_1 \cdot A([i+1]) \cdot D_1^{-1} = B([i+1])^T, \tag{16}$$

and show $B \overset{\text{DS}}{=} \text{ct}(A,\overline{S})$. Let D be an $n \times n$ invertible diagonal matrix defined as follows: For all $k \in [n]$,

$$D[k,k] = \begin{cases} D_1^{-1}[k,k] & \text{if } k \in [i] \\ \frac{D_2[k,k]}{D_2[i+1,i+1]} & \text{otherwise .} \end{cases}$$

We will prove that B is equal to $D \cdot \text{ct}(A,\overline{S}) \cdot D^{-1}$. Since S is a cut, the matrix A has the following structure.

$$A = \begin{array}{c} \\ S \\ \\ \overline{S} \end{array} \begin{array}{c} \begin{array}{cc} S & \overline{S} \end{array} \\ \begin{pmatrix} A(S) & A[S,i+1] \cdot \frac{A[i,\overline{S}]}{A[i,i+1]} \\ & \\ A[\overline{S},i] \cdot \frac{A[i+1,S]}{A[i+1,i]} & A(\overline{S}) \end{pmatrix} \end{array}$$

.

Thus, $\mathrm{ct}(A,\overline{S})$ can be written as follows.

$$\mathrm{ct}(A,\overline{S}) = \begin{array}{c} \\ S \\ \overline{S} \end{array}\begin{array}{c} \begin{array}{cc} \quad S \quad & \quad \overline{S} \quad \end{array} \\ \begin{pmatrix} A(S) & A[i+1,S]^T \cdot \frac{A[i,\overline{S}]}{A[i,i+1]} \\ A[\overline{S},i] \cdot \frac{A[S,i+1]^T}{A[i+1,i]} & A(\overline{S}) \end{pmatrix} \end{array}.$$

From Eq. (12) and Eq. (16), we have that

$$\begin{aligned} B[i,i+1] &= A[i,i+1] \cdot D_2^{-1}[i+1,i+1] \\ B[S,i+1] &= D_1^{-1}(S) \cdot A[i+1,S]^T \\ B[i,\overline{S}] &= A[i,\overline{S}] \cdot D_2^{-1}(\overline{S}). \end{aligned}$$

Using the above equation and Eq. (15),

$$\begin{aligned} B[S,\overline{S}] &= D_1^{-1}(S) \cdot A[i+1,S]^T \cdot \frac{D_2[i+1,i+1] \cdot A[i,\overline{S}] \cdot D_2^{-1}(\overline{S})}{A[i,i+1]} \\ &= D(S) \cdot \mathrm{ct}(A,\overline{S})[S,\overline{S}] \cdot D^{-1}(\overline{S}) \end{aligned}$$

Similarly, we can show that

$$B[\overline{S},S] = D(\overline{S}) \cdot \mathrm{ct}(A,\overline{S})[\overline{S},S] \cdot D^{-1}(S).$$

Applying Eq. (16) and Eq. (12), we get that

$$\begin{aligned} B(S) &= D(S) \cdot \mathrm{ct}(A,\overline{S})(S) \cdot D^{-1}(S), \text{ and} \\ B(\overline{S}) &= D(\overline{S}) \cdot \mathrm{ct}(A,\overline{S})(\overline{S}) \cdot D^{-1}(\overline{S}). \end{aligned}$$

Thus, $B = D \cdot \mathrm{ct}(A,\overline{S}) \cdot D^{-1}$. □

Lemma 3.8. *Let A be an $n \times n$ irreducible matrix over a field $\mathbb{F}$. Then, we can test whether A has a cut in $\mathsf{poly}(n)$ time. Moreover, if there exists a cut in A, then a minimal cut of A can be computed using $\mathsf{poly}(n)$ time.*

Proof. Let $2^{[n]}$ denote the set of all subsets of $[n]$. We first show that the functions $g_1, g_2 : 2^{[n]} \to \mathbb{Z}$, defined as

$$\forall X \in 2^{[n]},\ g_1(X) := \mathrm{rank}(A[X,\overline{X}]) \text{ and } g_2(X) := \mathrm{rank}(A[\overline{X},X]),$$

are submodular functions. For each $i \in [n]$, let V_i be the subspace of $\mathbb{F}^n$ spanned by the ith row vector of A and the characteristic vector χ_i for the set $\{i\}$. Let $f : 2^{[n]} \to \mathbb{Z}$ be the function defined as

$$\forall X \in 2^{[n]},\ f(X) = \dim\left(\sum_{e \in X} V_e\right).$$

It is not hard to verify that the function f is a submodular function. Observe that a subset of row vectors of $A[X,\overline{X}]$ indexed by $T \subseteq X$ are linearly independent if and only if the set $\{\chi_e \mid e \in X\} \sqcup \{A[e',[n]] \mid e' \in T\}$ are linearly independent. Therefore, for all $X \in 2^{[n]}$,

$$f(X) = g_1(X) + |X|.$$

Since f is a submodular function, g_1 is a submodular function. Similarly, we can show that g_2 is also a submodular function.

Since g_1 and g_2 are submodular functions, their sum $g = g_1 + g_2$ is also a submodular function. For any set $T = \{t_1, t_2\} \sqcup \{t_3, t_4\}$ with four distinct elements from $[n]$, let g_T be a function defined on subsets of $\overline{T}$ such that

$$\forall X \subseteq \overline{T},\ g_T(X) = g(X \cup \{t_1, t_2\}).$$

For any $X \subseteq \overline{T}$ and $a, b \in \overline{T}$,

$$\begin{aligned} g_T(X \cup \{a\}) + g_T(X \cup \{b\}) &= g(X \cup \{a, t_1, t_2\}) + g(X \cup \{b, t_1, t_2\}) \\ &\geq g(X \cup \{t_1, t_2\}) + g(X \cup \{a, b, t_1, t_2\}) \ \ (\text{ submodularity of } g) \\ &= g_T(X) + g_T(X \cup \{a, b\}). \end{aligned}$$

From the above, g_T is a submodular function. Note that if there exists a cut S in A with $\{t_1, t_2\} \subseteq S$ and $\{t_3, t_4\} \subseteq \overline{S}$ if and only if the minimum value of function g_T is at most 2. One can also observe that for any subset $X \subseteq \overline{T}$, $g_T(X)$ can be computed in $\mathsf{poly}(n)$ time. Thus, using the submodular minimization algorithm in [Sch03, Chapter 45], we can compute the minimum the value of g_T for any set $T = \{t_1, t_2\} \sqcup \{t_3, t_4\}$ of four distinct elements from $[n]$ in $\mathsf{poly}(n)$ time. There are at most n^4 such subsets T, and we can test whether A has a cut by computing the minimum value of g_T for all such possible subsets T. Thus, we can test whether A has a cut in $\mathsf{poly}(n)$ time.

Now, we discuss how to find a minimal cut. For a subset $T = \{t_1, t_2\} \sqcup \{t_3, t_4\}$ with four distinct elements from $[n]$, let g'_T be the function on subsets of $\overline{T}$ such that

$$\forall X \subseteq \overline{T},\ g'_T(X) = (n+1) g_T(X) + |X|.$$

Since both g_T and the cardinality function are submodular, g'_T is also a submodular function. Next observe that for $X \subseteq \overline{T}$, the set X minimizes g'_T if and only if for any $S \subseteq [n]$ with $t_1, t_2 \in S$ but $t_3, t_4 \notin S$ the following holds:

1. $g(X \cup \{t_1, t_2\}) \leq g(S)$.
2. if $g(X \cup \{t_1, t_2\}) = g(S)$, then $|X \cup \{t_1, t_2\}| \leq |S|$.

Therefore, a minimizing set of g'_T gives a minimal cut that contains both t_1 and t_2 but not t_3 and t_4, if such a cut exists. Now, using [Sch03, Theorem 45.1], we can compute minimizing sets for the submodular functions g'_T for all possible subsets T, and thus, we get a minimal cut in $\mathsf{poly}(n)$ time if A has a cut. □

Algorithm 1 Algorithm to test equal corresponding principal minors of two irreducible matrices

Input: Two $n \times n$ irreducible matrices A and B over $\mathbb{F}$

Output: If $A \overset{\text{PME}}{=} B$, then returns a sequence $\mathcal{X}$ of subsets of $[n]$ such that A, B and $\mathcal{X}$ satisfy the property $\mathcal{P}$. Otherwise, returns "No".

Using Claim 3.9, get D and $A' \leftarrow (A+D)^{\text{adj}}$ and $B' \leftarrow (B+D)^{\text{adj}}$.
$\mathcal{X} \leftarrow$ Finding-Cut-Sequence(A', B', $[n]$)
if $\mathcal{X} =$ "No" **then**
    return "No".
Let $\mathcal{X} = (X_1, X_2, \ldots, X_k)$.
return $(X_1, X_2, \ldots, X_k)$.

function Finding-Cut-Sequence(A, B, I)
    if $|I| \leq 3$, *or*, A has *no* cut **then**
        if A is *not* diagonally equivalent to B or B^T **then**
            return "No".
        else
            return I if $A \overset{\text{DS}}{=} B$, otherwise **return** $\emptyset$.
    else
        $\widetilde{B} \leftarrow B$
        Using Lemma 3.8, find a minimal cut $S \subseteq I$ in A.
        if $|S| \geq 3$, *and*, S is *not* a cut of B **then**
            return "No".
        else if $|S| = 2$, *and*, S is *not* a cut of B **then**
            $X \leftarrow$ Min-Cut-size-Two(A, B, S, I)
            if $X =$ "No" **then**
                return "No".
            $\widetilde{B} \leftarrow \text{ct}(B, X)$
        Let $s \in S$.
        $\mathcal{X}' \leftarrow$ Finding-Cut-Sequence($A(\overline{S}+s)$, $\widetilde{B}(\overline{S}+s)$, $\overline{S}+s$).
        if $\mathcal{X}' =$ "No" **then**
            return "No".
        Let $\mathcal{X}' = (X'_1, X'_2, \ldots, X'_k)$.
        $A_0 \leftarrow A$.
        for $i = 1$ to k **do**
            if $s \in X'_i$ **then**
                $X_i \leftarrow X'_i \cup S$
            else
                $X_i \leftarrow X'_i$
            $A_i \leftarrow \text{ct}(A_{i-1}, X_i)$.
        if $A_k \overset{\text{DS}}{=} \widetilde{B}$ **then**
            $\mathcal{X} \leftarrow (X_1, X_2, \ldots, X_k)$
        else if $\text{ct}(A_k, \overline{S}) \overset{\text{DS}}{=} \widetilde{B}$ **then**
            $\mathcal{X} \leftarrow (X_1, X_2, \ldots, X_k, \overline{S})$
        else
            return "No".
        if $|S| = 2$, *and*, S is *not* a cut of B **then**
            $\mathcal{X} \leftarrow (\mathcal{X}, X)$.
        return $\mathcal{X}$.

3.2 A description of the algorithm

Algorithm 2 Function for handling $|S| = 2$ case in function Cut-transpose of Algorithm 1

function Min-Cut-size-Two(A, B, I, S)
    $P \leftarrow \emptyset$, and $Q \leftarrow \emptyset$
    Let $s \in S$.
    for $t \in I \setminus S$ **do**
        if $A(S+t) \stackrel{\text{DS}}{=} B(S+t)$ **then**
            $P \leftarrow P \cup \{t\}$.
        else if $A(S+t) \stackrel{\text{DS}}{=} B(S+t)^T$ **then**
            $Q \leftarrow Q \cup \{t\}$.
        else
            return "No".
    $X \leftarrow P \cup \{s\}$.
    if X is *not* a cut of B **then**
        return "No".
    else
        return X.

3.3 Proof of Correctness of Algorithm 1

Claim 3.9. *Let $\mathbb{F}$ be a field of size greater than $10n^5$. Let A and B be two $n \times n$ irreducible matrices over $\mathbb{F}$. Then, in $\mathsf{poly}(n)$ time, we can find a diagonal matrix $D \in \mathbb{F}^{n \times n}$ such that $A + D$ and $B + D$ are nonsingular and all entries of $(A+D)^{\mathrm{adj}}$ and $(B+D)^{\mathrm{adj}}$ are nonzero.*

Remark 3.10. When the size of the underlying field $\mathbb{F}$ is not greater than $10n^5$, we can construct an extension $\mathbb{K}$ of $\mathbb{F}$ such that $|\mathbb{K}| > 10n^5$ and work with the larger field $\mathbb{K}$. We can also construct such an extension $\mathbb{K}$ in time $\mathsf{poly}(n)$.

Proof. Given A and B, we need to construct the following two types of diagonal matrices over $\mathbb{F}$ in $\mathsf{poly}(n)$ time.

Type I: Find diagonal matrices D_A and D_B such that both $A + D_A$ and $B + D_B$ are nonsingular.

Type II: For all $i, j \in [n]$, find diagonal matrices $A_{i,j}$ and $B_{i,j}$ such that

$$(A + A_{i,j})^{\mathrm{adj}}[i,j] \neq 0 \text{ and } (B + B_{i,j})^{\mathrm{adj}}[i,j] \neq 0.$$

Before describing the construction of the above-mentioned diagonal matrices, we first discuss how to use them to get the diagonal matrix D as promised in the claim. Using univariate polynomial interpolation, we combine all the above-mentioned diagonal matrices to a single $n \times n$ diagonal matrix $\widetilde{D}$ as follows. Let T be a subset of $\mathbb{F}$ of size $2n^2 + 2$. Fix a bijection ϕ from T to the set of diagonal matrices

$$\{D_A, D_B\} \sqcup \{A_i,\, B_i \mid i \in [n]\} \sqcup \{A_{i,j}\, B_{i,j} \mid i \neq j \in [n]\}.$$

For each $i \in [n]$, let P_i be a univariate polynomial in y such that for each $e \in T$, $P_i(e) = \phi(e)[i,i]$. We can find P_i in $\mathsf{poly}(n)$ time using Lagrange interpolation such that its degree is at most $2n^2 + 1$. Then, the diagonal matrix $\widetilde{D}$ is defined as $\widetilde{D}[i,i] = P_i$ for all $i \in [n]$. Observe that for each $e \in T$, after substituting y by e in $\widetilde{D}$, we get $\phi(e)$. Thus, both $A + \widetilde{D}$ and $B + \widetilde{D}$ are nonsingular, and all entries of $(A + \widetilde{D})^{\mathrm{adj}}$ and $(B + \widetilde{D})^{\mathrm{adj}}$ are nonzero. In other words, the univariate polynomials $\det(A + \widetilde{D})$, $\det(B + \widetilde{D})$, $(A + \widetilde{D})^{\mathrm{adj}}[i,j]$ and $(B + \widetilde{D})^{\mathrm{adj}}[i,j]$ for each $i, j \in [n]$ are nonzero. Note that each of these polynomials has a degree at most $(2n^2 + 1) \times n$.

Now, we have found a matrix with univariate polynomials as its entries that satisfy the condition of our claim.

Now consider the polynomial

$$P = \det(A + \widetilde{D}) \cdot \det(B + \widetilde{D}) \cdot \prod_{i,j \in [n]} (A + \widetilde{D})^{\mathsf{adj}}[i,j] \cdot (B + D)^{\mathsf{adj}}[i,j].$$

From [Csa76, Ber84], we know that the determinant of an $n \times n$ matrix whose entries are univariate polynomials of at most $\mathsf{poly}(n)$ degree can be computed in $\mathsf{poly}(n)$ time. Thus, the polynomial P can be computed in time $\mathsf{poly}(n)$. The degree of P is at most $d = (2n^3 + n) \times (2n^2 + 2) \leq 10n^5$. Therefore, for any subset $S \subseteq \mathbb{F}$ of size $d + 1$, there exists an $a \in S$ such that $P(a)$ is nonzero. Find such a point a in S. Given the polynomial P, this can be done in $\mathsf{poly}(n)$ time. This implies that all the polynomials in the product are also nonzero at $a \in S$. Hence, after substituting y by a in $\widetilde{D}$, we get a matrix D that satisfies the condition of our claim. Next, we describe how to find diagonal matrices of **Type I** and **Type II** in $\mathsf{poly}(n)$ time.

Find Type I diagonal matrices. Let y be an indeterminate and D' be a diagonal matrix with each diagonal entry is y. Then, the coefficient of y^n in $\det(A + D')$ is one, hence, $\det(A + D')$ is nonzero. Compute the polynomial $\det(A + D')$. Since it is an univariate polynomial of degree n, in $\mathsf{poly}(n)$time, we can find a point $a \in \mathbb{F}$ such that the evaluation of $\det(A + D')$ at a is nonzero. Then, the matrix D_A we get by substituting $y = a$ in D'. Similarly, we can find D_B in $\mathsf{poly}(n)$ time.

Finding Type II diagonal matrices. Let G_A be the graph such that its vertex set in $[n]$ and (i,j) is an edge inf G_A if and only if $i \neq j$ and $A[i,j] \neq 0$. Let $i,j \in [n]$. Since A is irreducible, there exists a path from i to j in G_A. Let

$$P = (i_0, i_1, i_2, \ldots, i_k) \text{ with } i_0 = i,\ i_k = j,$$

be a shortest path from i to j. In particular, when $i = j$, P is $(i_0 = i)$. We can compute such a path P in time $\mathsf{poly}(n)$. Let D' be a $n \times n$ diagonal matrix and y be an indeterminate such that for all $e \in [n]$,

$$D'[e,e] = \begin{cases} 0 & \text{if } e \in P \setminus \{i\} \\ y & \text{otherwise.} \end{cases}$$

Next, following the proof of [HL84, Theorem 1], one can show that

$$(A + D')^{\mathsf{adj}}[i,j] = \det\left((A + D')\left[[n] - j, [n] - i\right]\right) \neq 0.$$

From [Csa76, Ber84], we can compute $(A + D')^{\mathsf{adj}}[i,j]$ in time $\mathsf{poly}(n)$. It is a polynomial of degree at most $n - 1$. Therefore, in $\mathsf{poly}(n)$ time, we can find a point $a \in \mathbb{F}$ such that the evaluation of $(A + D')^{\mathsf{adj}}[i,j]$ at $y = a$ is nonzero. Then, the diagonal matrix $A_{i,j}$ we get by substituting $y = a$ in D'. Similarly, we find $B_{i,j}$ for all $i,j \in [n]$. □

Lemma 3.11. *Let A and B be two matrices over $\mathbb{F}$ such that their rows and columns are indexed by elements in I. Let the off-diagonal entries of A and B be nonzero. Then, given (A, B, I) as input to the function* FINDING-CUT-SEQUENCE *in Algorithm 1, it does the following:*

1. *If $A \overset{\text{PME}}{=} B$, then it returns a sequence $\mathcal{X}$ of less than $2|I|$ many subsets of I such that A, B and $\mathcal{X}$ satisfy the property $\mathcal{P}$.*

2. *Otherwise, it returns "No".*

Proof. We use induction to prove the above lemma.

Base case. The base case of our induction is either $|I| \leq 3$, or A has no cut. From Lemma 2.9, if $|I| \leq 3$ or A has no cut, then $A \overset{\text{PME}}{=} B$ if and only if either $A \overset{\text{DS}}{=} B$ or $A \overset{\text{DS}}{=} B^T$. Therefore, for the base case, the function Finding-Cut-Sequence in Algorithm 1 returns "No" when $A \overset{\text{PME}}{\neq} B$. Otherwise, it returns the sequence $\mathcal{X} = (I)$ when $A \overset{\text{DS}}{=} B$, and $\mathcal{X} = (\emptyset)$ when $A \overset{\text{DS}}{=} B^T$. Using Definition 2.11, $\mathcal{X}$ produces the matrix sequence (A, A) when $\mathcal{X} = (I)$, and produces the matrix sequence (A, A^T) when $\mathcal{X} = (\emptyset)$. Hence, for the base case, A, B and $\mathcal{X}$ satisfies the property $\mathcal{P}$.

Inductive step. In Line 16, the function Finding-Cut-Sequence (in Algorithm 1) computes a minimal cut S in the matrix A. If $|S| \geq 3$ and $A \overset{\text{PME}}{=} B$, then from Lemma 3.6, S is also a cut in B. This implies that if S is not a cut in B, then $A \overset{\text{PME}}{\neq} B$. Therefore, when $|S| \geq 3$ and S is not a cut of B, the function Finding-Cut-Sequence returns "No".

Now, consider the case when the size of the minimal cut S is two, but it is not a cut in B. Then, Algorithm 1 calls the function Min-cut-size-Two of Algorithm 2. It returns "No" when either a minor corresponding to set $S + t$ where $t \in I \setminus S$ is not same for A and B or when X, defined in Line 11, is not a cut of B. If a minor is not the same, then obviously $A \overset{\text{PME}}{\neq} B$. Otherwise from Lemma 3.3, $A \overset{\text{PME}}{\neq} B$ when X is not a cut of B. If $A \overset{\text{PME}}{=} B$, then from Lemma 3.3 X is a cut of B such that S is a cut of $\text{ct}(B, X)$.

Note that $\widetilde{B}$ is initially assigned to B at Line 15 of Algorithm 1. If the function Min-cut-size-Two in Algorithm 2 returns a cut X, the $\widetilde{B}$ is reassigned to $\text{ct}(B, X)$ at Line 23 of Algorithm 1. Thus, at the end of Line 23 of Algorithm 1, we have two matrices A and $\widetilde{B}$ such that S is a common cut of them, and also S is a minimal cut in A.

Let $s \in S$, $M = A(\overline{S} + s)$, and $N = \widetilde{B}(\overline{S} + s)$. Since the cardinality of S is at least two, the size of $\overline{S} + s$ is less than $|I|$. Therefore, from the induction hypothesis, the function Finding-Cut-Sequence on input $(M, N, \overline{S} + s)$ returns $\mathcal{X}'$ as follows:

1. If $M \overset{\text{PME}}{=} N$, then $\mathcal{X}' = (X'_1, X'_2, \ldots, X'_k)$ such that $k < 2|\overline{S} + s|$, $X'_i \subseteq \overline{S} + s$, and $\mathcal{X}'$ produces a sequence of matrices $(M = M_0, M_1, M_2, \ldots, M_k)$ satisfying the following:

$$\forall i \in [k],\ M_i = \text{ct}(M_{i-1}, X'_i) \text{ where } X'_i \text{ is } \emptyset, \overline{S} + s, \text{ or a cut in } M_{i-1}, \text{ and } M_k \overset{\text{DS}}{=} N. \quad (17)$$

2. Otherwise, $\mathcal{X}' =$"No".

If $M \overset{\text{PME}}{\neq} N$, then $A \overset{\text{PME}}{\neq} \widetilde{B}$. Applying Lemma 2.14, $A \overset{\text{PME}}{\neq} \widetilde{B}$ implies that $A \overset{\text{PME}}{\neq} B$. Therefore, when $\mathcal{X}' =$"No", the function Finding-Cut-Sequence also returns "No".

Now assume that $M \overset{\text{PME}}{=} N$, and $\mathcal{X}' = (X'_1, X'_2, \ldots, X'_k)$ satisfies Eq. (17). Let $(X_1, X_2, \ldots, X_k)$ be the sequence of subsets of I defined by the 'for loop' in Line 41 of Algorithm 1. From this sequence of subsets, the function Finding-Cut-Sequence defines a sequence of matrices $(A = A_0, A_1, A_2, \ldots, A_k)$ such that $A_i = \text{ct}(A_{i-1}, X_i)$ for all $i \in [k]$. To be well defined, this sequence of matrices should satisfy that X_i is a $\emptyset$, I or a cut in A_{i-1}. Next, we rely on the following claim, whose proof we defer to the end of the proof of this lemma.

Claim 3.12. *The sequence of matrices $(A_0 = A, A_1, A_2, \ldots, A_k)$ produced by the sequence $(X_1, X_2, \ldots, X_k)$ satisfies the following:*

1. *for all $i \in [k]$, X_i is a $\emptyset$, I or a cut in A_{i-1}.*
2. *for all $i \in [k]$, $M_i = A_i(\overline{S} + s)$ and S is a minimal cut in A_i.*

Using the above claim and Eq. (17), $A_k(\overline{S} + s) \overset{\text{DS}}{=} \widetilde{B}(\overline{S} + s)$. Applying Claim 3.12, S is also a minimal cut in A_k. Therefore, from Lemma 3.7, we have either $A_k \overset{\text{DS}}{=} \widetilde{B}$ or $\text{ct}(A_k, \overline{S}) \overset{\text{DS}}{=} \widetilde{B}$ when $A_k \overset{\text{PME}}{=} \widetilde{B}$. This implies that if $A_k \overset{\text{DS}}{\neq} \widetilde{B}$ and $\text{ct}(A_k, \overline{S}) \overset{\text{DS}}{\neq} \widetilde{B}$, then $A_k \overset{\text{PME}}{\neq} \widetilde{B}$, hence from Lemma 2.14, $A_k \overset{\text{PME}}{\neq} B$. Therefore, the function Finding-Cut-Sequence returns "No" when $A_k \overset{\text{DS}}{\neq} \widetilde{B}$ and

$\text{ct}(A_k, \overline{S}) \overset{\text{DS}}{\neq} \widetilde{B}$. When $A_k \overset{\text{DS}}{=} \widetilde{B}$, the function FINDING-CUT-SEQUENCE defines $\mathcal{X}$ as $(X_1, X_2, \ldots, X_k)$, and when $\text{ct}(A_k, \overline{S}) \overset{\text{DS}}{=} \widetilde{B}$, it defines $\mathcal{X}$ as $(X_1, X_2, \ldots, X_k, \overline{S})$. Therefore, at the end of Line 41 in Algorithm 1, we have a sequence $\mathcal{X}$ of subsets such that A, $\widetilde{B}$ and $\mathcal{X}$ satisfy the property $\mathcal{P}$.

Note that if S is a minimal cut in A of size 2 and it is not a cut in B, then $\widetilde{B}$ is defined as $\text{ct}(B, X)$. This implies that $B = \text{ct}(\widetilde{B}, X)$. Therefore, in Line 43 of Algorithm 1, $\mathcal{X}$ is updated by appending X at its end. Thus, we finally have a sequence $\mathcal{X}$ of subsets of I such that A, B, and $\mathcal{X}$ satisfy the property $\mathcal{P}$.

From the induction hypothesis, we know the length of the sequence $\mathcal{X}'$ is less than $2|\overline{S} + s|$, which is at most $2(|I| - 1)$. Observe that at the inductive step, in comparison with $\mathcal{X}'$, the length of final sequence $\mathcal{X}$ is increased by at most 2. Therefore, the length $\mathcal{X}$ is less than $2|I|$. □

Proof of Claim 3.12. We use induction to prove Claim 3.12.

Base case ($i = 1$). Note that X'_1 is $\emptyset$, $\overline{S} + s$ or a cut in $A_0(\overline{S} + s)$. If X'_1 is $\emptyset$ or $\overline{S} + s$, then the definition of X_i ensures that it is also $\emptyset$ or $\overline{S} + s$, respectively. Otherwise, when $X'_1$is a cut in $A(\overline{S} + s)$, Lemma 3.5 implies that X_1 is also a cut in A_0.

Since $X'_1 \subseteq X_1$, $A_1 = \text{ct}(A_0, X_1)$ and $M_1 = \text{ct}(M_0, X'_1)$, it is not hard to see that $A_1(\overline{S} + s) = M_1$. For the sake of contradiction assume that S is not a minimal cut in A_1. Then, there exists a cut S' in A_1 such that $S' \subset S$ but $S' \neq S$. From the definition of X_1, S is a subset of either X_1 or $\overline{X_1}$. Therefore, S' is also a subset of either X_1 or $\overline{X_1}$. Then, using Theorem 3.1, S' is also a cut in A_0. This is a contradiction, since S is a minimal cut in A_0.

Inductive step ($i > 1$). From the induction hypothesis, S is a minimal cut in A_{i-1} and $A_{i-1}(\overline{S} + s) = M_{i-1}$. We also have that X'_i is $\emptyset$, $\overline{S} + s$ or a cut in M_{i-1}. Now, using analysis similar to the base case, we can prove our inductive step. □

Theorem 3.13. *Let A and B be two irreducible $n \times n$ matrices over a field $\mathbb{F}$. Then, given A and B as input to Algorithm 1, it does the following:*

1. *If $A \overset{\text{PME}}{=} B$, it returns a sequence $\mathcal{X}$ of less than $2n$ many subsets of $[n]$ such that A, B and $\mathcal{X}$ satisfy the property $\mathcal{P}$.*
2. *Otherwise, it returns "No".*

Proof. The input matrices A and B are irreducible. Therefore, using Claim 3.9, we have an $n \times n$ diagonal matrix D over $\mathbb{F}$ such that the matrices $A + D$ and $B + D$ are nonsingular and all the entries of the matrices $A' = (A + D)^{\text{adj}}$ and $B' = (B + D)^{\text{adj}}$ are nonzero. Assume that the rows and columns of A' and B' are indexed by $[n]$. From Lemma 3.11, given A', B' and $[n]$ as input to the function FINDING-CUT-SEQUENCE in Algorithm 1, it returns $\mathcal{X}$ satisfying the following property:

1. If $A' \overset{\text{PME}}{\neq} B'$, then $\mathcal{X} =$"No".
2. Otherwise, $\mathcal{X} = (X_1, X_2, \ldots, X_\ell)$ is a sequence of less than $2n$ many subsets of $[n]$ such that A', B' and $\mathcal{X}$ satisfy the property $\mathcal{P}$. That is, $\mathcal{X}$ produces a sequences of matrices $(A' = A'_0, A'_1, A'_2, \ldots, A'_\ell)$ such that

$$\forall i \in [\ell],\ A'_i = \text{ct}(A'_{i-1}, X_i) \text{ where } X_i \text{ is } \emptyset, [n], \text{ or a cut in } A'_{i-1}, \text{ and } A'_\ell \overset{\text{DS}}{=} B.$$

Since $A + D$ and $B + D$ are nonsingular, from Lemma 2.1, $A \overset{\text{PME}}{=} B$ if and only if $A' \overset{\text{PME}}{=} B'$. Therefore, Algorithm 1 returns "No" when $A \overset{\text{PME}}{\neq} B$. Next, consider the case when $A' \overset{\text{PME}}{=} B'$, which is equivalent to $A \overset{\text{PME}}{=} B$. Then $\mathcal{X} = (X_1, X_2, \ldots, X_\ell)$. Using induction, we now show the following claim.

Claim 3.14. *The sequence of $\mathcal{X} = (X_1, X_2, \ldots, X_\ell)$ of subsets of $[n]$ produces a sequence of matrices $(A = A_0, A_1, A_2, \ldots, A_\ell)$ such that $A_i = \mathrm{ct}(A_{i-1}, X_i)$ and $(A_i + D)^{\mathrm{adj}} \overset{\mathrm{DS}}{=} A_i'$ for all $i \in [\ell]$.*

Base case $(i = 1)$**.** From the definition, $A_0' = (A_0 + D)^{\mathrm{adj}}$. We know that X_1 is $\emptyset$, $[n]$, or a cut of A_0'. Using Lemma 2.7, if X_1 is a cut in A_0', then X_1 is also a cut of A_0. Therefore, $A_1 = \mathrm{ct}(A_0, X_1)$ is well defined. Applying Lemma 2.15, we also have that $(A_1 + D)^{\mathrm{adj}} \overset{\mathrm{DS}}{=} A_1'$. Hence, the base case is proved.

Inductive step $(i > 1)$**.** From the inductive hypothesis, we have that $(A_{i-1} + D)^{\mathrm{adj}} \overset{\mathrm{DS}}{=} A_{i-1}'$. The rest of the proof in the inductive step is similar to the base case

This completes the proof of Claim 3.14. Now, we have a sequence $\mathcal{X} = (X_1, X_2, \ldots, X_\ell)$ of less than $2n$ many subsets such that it produces a sequence of matrices $(A = A_0, A_1, A_2, \ldots, A_\ell)$ with the following property:

$$\forall i \in [\ell],\ A_i = \mathrm{ct}(A_{i-1}, X_i) \text{ where } X_i \text{ is } \emptyset, [n], \text{ or a cut in } A_{i-1}, \text{ and } (A_\ell + D)^{\mathrm{adj}} \overset{\mathrm{DS}}{=} A_\ell'.$$

Since $A_\ell' \overset{\mathrm{DS}}{=} B'$, using Lemma 2.10, $A_\ell \overset{\mathrm{DS}}{=} B$. Thus, A, B and $\mathcal{X}$ satisfy the property $\mathcal{P}$ when $A \overset{\mathrm{PME}}{=} B$. This completes the proof of our theorem. □

3.4 Time complexity of Algorithm 1

Now, we analyze the time complexity of Algorithm 1. More specifically, we show that given two $n \times n$ irreducible matrices over $\mathbb{F}$ as input, Algorithm 1 performs $\mathsf{poly}(n)$ $\mathbb{F}$-operations. To achieve this, we first discuss the complexity of the function Finding-Cut-Sequence in Algorithm 1. In particular, we prove that given (A, B, I) as input, the function Finding-Cut-Sequence performs $\mathsf{poly}(|I|)$ $\mathbb{F}$-operations.

Time complexity of the function Finding-Cut-Sequence. Let $T(m)$ be the number of $\mathbb{F}$-operations performed by the function Finding-Cut-Sequence when the size of the input matrices A and B is $m \times m$, i.e., $|I| = m$. We show that $T(m)$ is at most $\mathsf{poly}(m)$. First, when $|I| \leq 3$ or A has no cute, the function checks whether $A \overset{\mathrm{DS}}{=} B$ or B^T. From the **??**, this can be done in $\mathsf{poly}(m)$ $\mathbb{F}$-operations. Therefore, if the input matrix A has no cut,

$$T(m) \leq \mathsf{poly}(m).$$

Now, when A has a cut, the function first computes a minimal cut S in A. Lemma 3.8 ensures that we can find S in $\mathsf{poly}(m)$ many $\mathbb{F}$-operations. Then, if $|S| \geq 3$, it checks whether S is also a cut in B. Using the observation 2.6, this can be verified in $\mathsf{poly}(m)$ $\mathbb{F}$-operations. If $|S| \geq 3$ and it is also not a cut in B, it returns "No". Hence, in that case,

$$T(m) \leq \mathsf{poly}(m).$$

Next, consider the size of the minimal cut S is two. Then, the function Finding-Cut-Sequence needs to check whether S is also a cut in B. Again, from the observation 2.6, this can be done in $\mathsf{poly}(m)$ $\mathbb{F}$-operations. If S is not a cut in B, the function calls another function Min-Cut-size-Two in Algorithm 2 with the input (A, B, I, S).

The function Min-Cut-size-Two computes a partition $P \sqcup Q$ of the set $I \setminus S$. To do this, for all $t \in I \setminus S$, the function Min-Cut-size-Two needs to check whether $A(S + t) \overset{\mathrm{DS}}{=} B(S + t)$ or $B(S + t)^T$. For some $t \in I \setminus S$, $A(S + t) \overset{\mathrm{DS}}{\neq} B(S + t)$ and $B(S + t)^T$ implies that $A \overset{\mathrm{PME}}{\neq} B$, hence the function Min-Cut-size-Two returns "No". Otherwise, it successfully computes the partition $P \sqcup Q$ of $I \setminus S$. Since the size of $S + t$ is three, applying the Claim 2.8, verifying whether $A(S+t) \stackrel{\text{DS}}{=} B(S+t)$ or $B(S+t)^T$ can be done in constant $\mathbb{F}$-operations. Therefore, computing the partition $P \sqcup Q$ takes $O(|I \setminus S|)$ many $\mathbb{F}$-operations. Next, the function Min-Cut-size-Two checks whether $X = P \cup \{s\}$ is cut of B or not, and from the observation 2.6, this can be done in $\mathsf{poly}(m)$ $\mathbb{F}$-operations. If X is also a cut in B, the function Min-Cut-size-Two returns X, otherwise "No". Overall, if the function Min-Cut-size-Two returns "No", the function Finding-Cut-Sequence also returns "No". Thus, when the function Min-Cut-size-Two returns "No",

$$T(m) \leq \mathsf{poly}(m).$$

If X is a cut of B, then the function Finding-Cut-Sequence requires to compute the matrix $\mathrm{ct}(B, X)$. From observation 2.13, the matrix $\mathrm{ct}(B, X)$ can be computed in $\mathsf{poly}(m)$ $\mathbb{F}$-operations. After Line 23 in Algorithm 1, we have two matrices A and $\widetilde{B}$ which have a common cut S. Then, the function Finding-Cut-Sequence makes a recursive call with the input $(A(\overline{S}+s), B(\overline{S}+s), \overline{S}+s)$. Using induction, this recursive call performs $T(m')$ many $\mathbb{F}$-operations where $m' = |\overline{S}+s|$. If the return value $\mathcal{X}'$ of this recursive call is "No", the function Finding-Cut-Sequence also returns "No". Thus, in that case,

$$T(m) = T(m') + \mathsf{poly}(m).$$

Now assume that the return value $\mathcal{X}'$ is not "No". Then, $\mathcal{X}' = (X'_1, X'_2, \ldots, X'_k)$ is a sequence of subsets of $\overline{S}+s$. Next, the function Finding-Cut-Sequence extends it to a sequence $\mathcal{X} = (X_1, X_2, \ldots, X_k)$ of subsets of I. As explained in the proof of Lemma 3.11, $\mathcal{X}$ produces a sequence of matrices $(A = A_0, A_1, A_2, \ldots, A_k)$ such that $A_i = \mathrm{ct}(A_{i-1}, X_i)$ where $i \in [k]$. From observation 2.13, in Line 35 of Algorithm 1, computing A_i for each $i \in [k]$ needs at most $\mathsf{poly}(m)$ $\mathbb{F}$-operations. Therefore, to compute the matrix A_k from $A_0 = A$, the total number of $\mathbb{F}$-operations required is at most $\mathsf{poly}(m)$. Then, the function Finding-Cut-Sequence needs to verify whether $\widetilde{B} \stackrel{\text{DS}}{=} A_k$ or $\mathrm{ct}(A_k, S)$. From the **??** and observation 2.13, verifying whether $\widetilde{B} \stackrel{\text{DS}}{=} A_k$ or $\mathrm{ct}(A_k, S)$ needs at most $\mathsf{poly}(m)$ $\mathbb{F}$-operations. If $\widetilde{B} \stackrel{\text{DS}}{\neq} A_k$ and $\mathrm{ct}(A_k, S)$, the function Finding-Cut-Sequence returns "No", otherwise it updates $\mathcal{X}$ appropriately and returns it. These updates can be done efficiently.

Thus, from the above discussion, we can conclude that

$$T(m) \leq T(m') + \mathsf{poly}(m), \text{ where } m' < m.$$

This implies that $T(m)$ is at most $\mathsf{poly}(m)$.

Time complexity of Algorithm 1. Now, we discuss the time complexity of Algorithm 1. Given two $n \times n$ irreducible matrices A and B over $\mathbb{F}$, Algorithm 1 first computes a diagonal matrix D such that both $A + D$ and $B + D$ are nonsingular and all the entries of $A' = (A+D)^{\mathrm{adj}}$ and $B' = (B+D)^{\mathrm{adj}}$ are nonzero. Claim 3.9 ensures that we can compute such a diagonal matrix D in $\mathsf{poly}(n)$ $\mathbb{F}$-operations. Then Algorithm 1 calls the function Find-Cut-Sequence with input $(A, B, [n])$. From the above discussion, the function Find-Cut-Sequence performs at most $\mathsf{poly}(n)$ many $\mathbb{F}$-operations. Thus, Algorithm 1 performs at most $\mathsf{poly}(n)$ $\mathbb{F}$-operations.

3.5 Proof of Theorem 1.1

Now, we discuss the proof of Theorem 1.1.

Proof of Theorem 1.1. First, assume that there exists a sequence $\mathcal{X} = (X_1, X_2, \ldots, X_\ell)$ of subsets of $[n]$ such that it produces a sequence of matrices $(A = A_0, A_1, A_2, \ldots, A_\ell)$ with the following property:

$$\forall i \in [n],\ A_i = \mathrm{ct}(A_{i-1}, X_i) \text{ where } X_i \text{ is } \emptyset, [n], \text{ or a cut in } A_{i-1}, \text{ and } A_\ell \stackrel{\text{DS}}{=} B.$$

If $X_i = \emptyset$ or $[n]$ for some $i \in [\ell]$, then it is easy to see that $A_{i-1} \stackrel{\text{PME}}{=} A_i$. From Lemma 2.14, if X_i is a cut of A_{i-1} for some $i \in [\ell]$, then $A_{i-1} \stackrel{\text{PME}}{=} A_i$. Thus, $A \stackrel{\text{PME}}{=} B$ when A, B and $\mathcal{X}$ satisfy $\mathcal{P}$. The converse direction follows from Theorem 3.13. From the discussion in Section 3.4, we know that in $\mathsf{poly}(n)$ $\mathbb{F}$-operations, a sequence $\mathcal{X}$ such that A, B and $\mathcal{X}$ satisfy the property $\mathcal{P}$ can be computed when $A \stackrel{\text{PME}}{=} B$. □

4 Proof of Theorem 1.2: Testing Principal Minor Equivalence

4.1 A description of the algorithm

Algorithm 3 Algorithm to test equal corresponding principal minors of two matrices

Input: Two $n \times n$ matrices A and B over $\mathbb{F}$ whose rows and columns are indexed by $[n]$
Output: Output "Yes" if $A \stackrel{\text{PME}}{=} B$, otherwise output "No"

```
Compute the directed graph G_A with the vertex set [n] and (i, j) is an edge of G_A if and
  only if i ≠ j and A[i, j] ≠ 0.
Compute the directed graph G_B with the vertex set [n] and (i, j) is an edge of G_B if and only
  if i ≠ j and B[i, j] ≠ 0.
Compute the strongly connected components of G_A and G_B.
Let I_1, I_2, ... I_s be the strongly connected components of G_A.
Let I'_1, I'_2, ... I'_{s'} be the strongly connected components of G_B.
if {I_1, I_2, ..., I_s} ≠ {I'_1, I'_2, ..., I'_s} then
    Output "No".
else
    for i = 1 to s do
        ans ← output of Algorithm 1 with input matrices A(I_i) and B(I_i).
        if ans = "No" then
            Output "No", and stop execution.
    Ouput "Yes".
```

4.2 Analysis of Algorithm 3

Proof of Theorem 1.2. We start with a proof of correctness of Algorithm 3.

Correctness of Algorithm 3. The rows and the columns of the input matrices A and B are indexed by $[n]$. In Algorithm 3, we have two directed graphs G_A and G_B. From the observation 2.3, after permuting the rows and the corresponding columns, the matrix A can be written as a block upper triangular matrix such that the diagonal blocks are $A(I_1), A(I_2), \ldots, A(I_s)$ and each $A(I_i)$ is a irreducible matrix. Similarly, after permuting the rows and the corresponding columns, B can be written as written as block upper triangular matrix such that the diagonal blocks are $B(I'_1), B(I'_2), \ldots, B(I'_{s'})$ and each $B(I'_i)$ is an irreducible matrix. Then, from Lemma 2.4, if $\{I_1, I_2, \ldots, I_s\} \neq \{I'_1, I'_2, \ldots, I'_{s'}\}$, then $A \stackrel{\text{PME}}{\neq} B$. Therefore, Algorithm 3 returns "No" when $\{I_1, I_2, \ldots, I_s\} \neq \{I'_1, I'_2, \ldots, I'_{s'}\}$.

Now consider when $\{I_1, I_2, \ldots, I_s\} = \{I'_1, I'_2, \ldots, I'_{s'}\}$. Then, $s = s'$. Again using Lemma 2.4, $A \stackrel{\text{PME}}{=} B$ if and only $A(I_i) \stackrel{\text{PME}}{=} B(I_i)$ for all $i \in [s]$. From Theorem 3.13, given two irreducible matrices as input to Algorithm 1, it returns "No" if and only if the corresponding principal minors of the input matrices are not equal. Thus, Algorithm 3 outputs "No" if $A \stackrel{\text{PME}}{\neq} B$, otherwise outputs "Yes".

Time complexity of Algorithm 3. From the definition, given the matrices A and B, the directed graphs G_A and G_B can be computed in time $\mathsf{poly}(n)$. Given the directed graphs G_A and G_B, we can compute its strongly connected component in time $\mathsf{poly}(n)$. From the time complexity analysis of Algorithm 1 (Section 3.4), we know each invocation of Algorithm 1 needs at most $\mathsf{poly}(n)$ time. Thus, the total time complexity of Algorithm 1 is at most $\mathsf{poly}(n)$. □

5 PIT for Sum of two DET1

In this section, we show Theorem 1.3. Given two sequences of $n \times n$ matrices $(A_0, A_1, \ldots, A_m)$ and $(B_0, B_1, \ldots, B_m)$ over a field $\mathbb{F}$ such that the rank of A_i and B_i is at most 1 for $1 \le i \le n$, the goal is to decide whether two polynomials $P_1 = \det(A_0 + A_1 y_1 + \ldots + A_m y_m)$ and $P_2 = \det(B_0 + B_1 y_1 + \ldots + B_m y_m)$ are the same in $\mathsf{poly}(m, n)$ $\mathbb{F}$-operations. First, we consider the case when A_0 and B_0 are the zero matrix. Then, we reduce the general case where there are no constraints on A_0 and B_0 to this case. Then, we give a polynomial time reduction from this problem to the problem of equivalence testing of principal minors of two $m \times m$ matrices. For integers p and q, let 0_p and $0_{p,q}$ denote the $p \times p$ and $p \times q$ matrix, respectively, with all zeros.

5.1 $A_0 = B_0 = 0_n$.

Let $A_j = u_{1,j} \cdot v_{1,j}^T$ and $B_j = u_{2,j} \cdot v_{2,j}^T$ for each $j \in [m]$ where $u_{1,j}, v_{1,j}, u_{2,j}, v_{2,j} \in \mathbb{F}^n$. Let U_i, V_i be $n \times m$ matrices such that their jth column are $u_{i,j}$ and $v_{i,j}$, respectively, for $i \in \{1, 2\}$ and $j \in [m]$. Let Y be an $m \times m$ diagonal matrix with indeterminate y_i as the ith diagonal entry. Then,

$$\det(A_1 y_1 + \ldots + A_m y_m) = \det(U_1 Y V_1^T) \text{ and } \det(B_1 y_1 + \ldots + B_m y_m) = \det(U_2 Y V_2^T).$$

For a subset T of $[m]$, let $y_T = \prod_{e \in T} y_e$, $U_{i,T} = U[[n], T]$ and $V_{i,T} = V[[n], T]$ for $i \in \{1, 2\}$. Using the Cauchy-Binet formula for multiplying two rectangular matrices,

$$\det(U_i Y V_i^T) = \sum_{T \subseteq [m], |T| = n} (\det(U_{i,T}) \det(V_{i,T}) y_T) \quad \text{for } i \in \{1, 2\}.$$

Hence, by comparing coefficients of monomials of P_1 and P_2, we get

$$P_1 = P_2 \iff \det(U_{1,T}) \det(V_{1,T}) = \det(U_{2,T}) \det(V_{2,T}) \;\; \forall T \subseteq [m] \text{ with } |T| = n. \tag{18}$$

Now, we discuss how to test the latter part mentioned above. First, we find a set T of size $[n]$ such that $\det(U_{1,T}) \det(V_{1,T})$ is non-zero using a matroid intersection algorithm for matroids represented by U_1 and V_1 in $\mathsf{poly}(m, n)$ $\mathbb{F}$-operations. If such T doesn't exist, then $P_1 = 0$. Similarly, we can check whether P_2 is zero and decide whether $P_1 = P_2$. Suppose such a set T exists and without loss of generality, let $T = [n]$. If $\det(U_{1,[n]}) \det(V_{1,[n]}) \ne \det(U_{2,[n]}) \det(V_{2,[n]})$, then $P_1 \ne P_2$ from Eq. (18).

Suppose $\det(U_{1,[n]}) \det(V_{1,[n]}) = \det(U_{2,[n]}) \det(V_{2,[n]})$. Now, we have to check this for other sets T of size n. Let $U'_i = U_{i,[n]}^{-1} \cdot U_i$ and $V'_i = V_{i,[n]}^{-1} \cdot V_i$ for $i = 1, 2$. Since $U_i = U_{i,[n]} \cdot U'_i, V_i = V_{i,[n]} \cdot V'_i$ for $i = 1, 2$ and $\det(U_{1,[n]}) \det(V_{1,[n]}) = \det(U_{2,[n]}) \det(V_{2,[n]})$, for any set T of size n,

$$\det(U_{1,T}) \det(V_{1,T}) = \det(U_{2,T}) \det(V_{2,T}) \iff \det(U'_{1,T}) \det(V'_{1,T}) = \det(U'_{2,T}) \det(V'_{2,T}) \tag{19}$$

Note that $U'_{i,[n]} = V'_{i,[n]} = I_n$. For $i = 1, 2$, let $\widehat{U}_i$ and $\widehat{V}_i$ be the $n \times (m - n)$ matrices defined as $\widehat{U}_i[[n], [m] \setminus [n]]$ and $\widehat{V}_i[[n], [m] \setminus [n]]$, respectively. For $i \in \{1, 2\}$ and a set $T = T'_1 \sqcup T'_2$ of size n with $T'_1 \subseteq [n], T'_2 \subseteq [m] - [n]$ such that $T'_2 = \{n + e \mid e \in T_2\}$ where $T_2 \subseteq [m - n]$,

$$\det(U'_{i,T}) = \sigma(T) \det(U'_i[[n] \setminus T'_1, T'_2]) \text{ and } \det(V'_{i,T}) = \sigma(T) \det(V'_i[[n] \setminus T'_1, T'_2]) \tag{20}$$

where $\sigma : \binom{[m]}{n} \to \{1,-1\}$ is some sign function on n sized subsets of $[m]$. Since $U_i'[[n]\setminus T_1', T_2'] = \widehat{U}_1[T_1,T_2]$ and $V_i'[[n]\setminus T_1', T_2'] = \widehat{V}_1[T_1,T_2]$ where $T_1 = [n]\setminus T_1'$, using Eqs. (18) to (20) we get

$$P_1 = P_2 \iff \det(\widehat{U}_1[T_1,T_2])\det(\widehat{V}_1[T_1,T_2]) = \det(\widehat{U}_2[T_1,T_2])\det(\widehat{V}_2[T_1,T_2]) \quad (21)$$
$$\text{for each } T_1 \subseteq [n], T_2 \subseteq [m-n] \text{ with } |T_1| = |T_2|$$

Let A and B be the $m\times m$ matrices defined as follows:

$$A = \left[\begin{array}{c|c} 0_{m-n} & \widehat{V}_1^T \\ \hline -\widehat{U}_1 & 0_n \end{array}\right] \quad\text{and}\quad B = \left[\begin{array}{c|c} 0_{m-n} & \widehat{V}_2^T \\ \hline -\widehat{U}_2 & 0_n \end{array}\right].$$

Let us consider the principal minors of A and B. If a set T is a subset of $[m-n]$ or $[m]-[m-n]$, then the corresponding principal minors of both A and B are zero. Consider a set $T = T_1' \sqcup T_2$ such that $T_2 \subseteq [m-n]$ and $T_1' \subseteq [m]-[m-n]$ such that $T_1' = \{m-n+e \mid e \in T_1\}$ where $T_1 \subseteq [n]$. Then,

$$A[T] = \left[\begin{array}{c|c} 0_{|T_2|} & \widehat{V}_1[T_1,T_2]^T \\ \hline -\widehat{U}_1[T_1,T_2] & 0_{|T_1|} \end{array}\right] \quad\text{and}\quad B[T] = \left[\begin{array}{c|c} 0_{|T_2|} & \widehat{V}_2[T_1,T_2]^T \\ \hline -\widehat{U}_2[T_1,T_2] & 0_{|T_1|} \end{array}\right].$$

Note that if $|T_1| \neq |T_2|$, then both $\det(A[T])$ and $\det(B[T])$ are zero. If $|T_1| = |T_2|$, then

$$\det(A[T]) = \det(\widehat{U}_1[T_1,T_2])\det(\widehat{V}_1[T_1,T_2]); \det(B[T]) = \det(\widehat{U}_2[T_1,T_2])\det(\widehat{V}_2[T_1,T_2]). \quad (22)$$

From above discussion and Eq. (22),

$$A \overset{\text{PME}}{=} B \iff \det(\widehat{U}_1[T_1,T_2])\det(\widehat{V}_1[T_1,T_2]) = \det(\widehat{U}_2[T_1,T_2])\det(\widehat{V}_2[T_1,T_2]) \quad (23)$$
$$\forall T_1 \subseteq [n], T_2 \subseteq [m-n] \text{ with } |T_1| = |T_2|.$$

From Eq. 21 and Eq. 23, $P_1 = P_2 \iff A \overset{\text{PME}}{=} B$. Note that A and B can be computed using $\mathsf{poly}(m,n)$ $\mathbb{F}$-operations. From Theorem 1.1, we can check whether $A \overset{\text{PME}}{=} B$ in $\mathsf{poly}(m)$ $\mathbb{F}$-operations. This completes the proof of Theorem 1.3 when A_0 and B_0 are the zero matrix.

5.2 No constraint on A_0 and B_0

From previous section, $P_1 = \det(A_0 + U_1 Y V_1^T)$ and $P_2 = \det(B_0 + U_2 Y V_2)^T$. From [**?**, Lemma 4.3] $P_1 = \det(C_1)$ and $P_2 = \det(C_2)$ such that

$$C_1 = \begin{pmatrix} I_m & Y & 0_{m,n} \\ 0_m & I_m & V_1^T \\ U_1 & 0_{n,m} & A_0 \end{pmatrix} \text{ and } C_2 = \begin{pmatrix} I_m & Y & 0_{m,n} \\ 0_m & I_m & V_2^T \\ U_2 & 0_{n,m} & A_0 \end{pmatrix}.$$

If we compute $P_i = \det(C_i)$ using the Generalized Laplace Theorem by fixing the first m rows, we get that P_i is multilinear and the coefficient of y_T for a subset T of $[m]$ is

$$\sigma(T)\det(C_i[[2m+n]\setminus[m], \phi(T)])$$

where σ is some sign function depending on set T and $\phi : 2^{[m]} \to \binom{[2m+n]}{m+n}$ such that $\phi(T) = T \cup \{e+m \mid e \notin T\} \cup ([2m+n]\setminus[2m])$. Hence,

$$P_1 = P_2 \iff \det(C_1[\overline{[m]}, \phi(T)]) = \det(C_2[\overline{[m]}, \phi(T)]) \ \forall T \subseteq [m]. \quad (24)$$

Let V be the following $(m+n) \times (2m+n)$ matrix $\begin{pmatrix} I_m & I_m & 0_{m,n} \\ 0_{n,m} & 0_{n,m} & I_n \end{pmatrix}$. For any set $T' \subset [2m+n]$ of size $m+n$ which is not in image of ϕ, $\det(V[[m+n], T']) = 0$ and if T' belongs to image of map ϕ, then $\det(V[[m+n], T']) = 1$. Hence,

$$\det(C_1[\overline{[m]}, \phi(T)]) = \det(C_2[\overline{[m]}, \phi(T)]) \ \forall T \subseteq [m] \iff$$
$$\det(C_1[\overline{[m]}, T']) \det(V_{T'}) = \det(C_2[\overline{[m]}, T']) \det(V_{T'}) \ \forall T' \in \binom{[2m+n]}{m+n}. \tag{25}$$

Note that the right-hand side of the above equation is similar to Eq. (18). Hence, using similar arguments from the previous section, checking the later part of Eq. 25 can be reduced to checking whether principal minors of two $(2m+n) \times (2m+n)$ matrices (that can be computed in $\mathsf{poly}(n)$ time) are the same. Hence, from Eq. (24), checking whether $P_1 = P_2$ reduces to checking whether principal minors of two $(2m+n) \times (2m+n)$ matrices are the same. This completes the proof of Theorem 1.3.

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A Missing Proofs from Section 2

Lemma 2.14 (restated). *Let A be an $n \times n$ matrix over a field $\mathbb{F}$. Let $X \subseteq [n]$ be a cut in A. Then, $A \overset{\scriptscriptstyle PME}{=} \mathrm{ct}(A, X)$.*

Proof. Since X is a cut in A, the matrix A can be written as follows:

$$A = \begin{array}{c} \\ X \\ \\ \overline{X} \end{array} \begin{array}{c} \begin{array}{cc} X & \overline{X} \end{array} \\ \begin{pmatrix} M & p \cdot q^T \\ & \\ u \cdot v^T & N \end{pmatrix} \end{array},$$

where $p, v \in \mathbb{F}^{|\overline{X}|}$ and $q, u \in \mathbb{F}^{|X|}$. Then, from Definition 2.11,

$$\mathrm{ct}(A, X) = \begin{array}{c} \\ X \\ \\ \overline{X} \end{array} \begin{array}{c} \begin{array}{cc} X & \overline{X} \end{array} \\ \begin{pmatrix} M & p \cdot u^T \\ & \\ q \cdot v^T & N^T \end{pmatrix} \end{array}.$$

Let $S \subseteq [n]$. Observe that if S is a subset of X or $\overline{X}$, then $\det(A(S)) = \det(\mathrm{ct}(A, X)(S))$. Now consider that $S = S_1 \sqcup S_2$ such that S_1 and S_2 are nonempty subsets of X and $\overline{X}$, respectively. Next, we prove that $\det(A(S)) = \det(\mathrm{ct}(A, X)(S))$.

Assume that the coordinates p, v are indexed by X and the coordinates of q, u are indexed by $\overline{X}$. By $p_{S_1}, q_{S_2}, u_{S_1}$ and v_{S_2}, we denote the projection of the respective vectors on the respective coordinates. Let $A' = A(S)$ and $B' = \mathrm{ct}(A, X)(S)$. Then,

$$A' = \begin{pmatrix} M(S_1) & p_{S_1} \cdot q_{S_2}^T \\ & \\ u_{S_2} \cdot v_{S_1}^T & N(S_2) \end{pmatrix}, \quad \text{and} \quad B' = \begin{pmatrix} M(S_1) & p_{S_1} \cdot u_{S_2}^T \\ & \\ q_{S_2} \cdot v_{S_1}^T & N(S_2)^T \end{pmatrix}.$$

If either of $p_{S_1}, q_{S_2}, u_{S_2}$, or v_{S_1} is the zero vector, then

$$\det(A') = \det(B') = \det(M(S_1)) \det(N(S_2)).$$

Next, assume that all of them are nonzero.

Let $\ell = |S|$, $k = |S_1|$, and $K = [k]$. Suppose that the rows and columns of A' and B' are indexed by $[\ell]$, and the rows and columns of $M(S_1)$ are indexed by K. For each $i \in K$, let M_i denote the $k \times k$ matrix obtained by removing ith column of $M(S_1)$ and appending p_{S_1} as the kth column. For $j \in \overline{K}$, let N_j denote the $(l - k) \times (l - k)$ matrix obtained by removing jth column of $N(S_2)$ and adding u_{S_2} as the first column. Using the Generalized Laplace Theorem (see [Ahm23, Theorem 3.1]), $\det(A')$ can be written as follows.

$$\det(A') = \sum_{T \subseteq [\ell], |T| = k} (-1)^{\Sigma K + \Sigma T} \det(A'[K, T]) \det(A'[\overline{K}, \overline{T}]).$$

Note that for all $T \subset [\ell]$ with $|T \cap \overline{K}| \geq 2$, the submatrix $A'[K, T])$ is not full rank since K is a cut for the matrix A'. Therefore, for all such $T \subseteq [\ell]$ with $|T| = k$, $\det(A'[K, T]) = 0$. This implies that

$$\det(A') = \det(M(S_1)) \det(N(S_2)) + \sum_{i \in K, j \in \overline{K}} (-1)^{j-i} \det(A'[K, K - i + j]) \det(A'[\overline{K}, \overline{K} + i - j]).$$

Observe that for all $i \in K$ and $j \in \overline{K}$,

$$\det(A'[K, K - i + j]) = q_{S_2}[j - k] \det(M_i), \text{and } \det(A[\overline{K}, \overline{K} + i - j]) = v_{S_1}[i] \det(N_j).$$

Therefore,

$$\det(A') = \det(M(S_1))\det(N(S_2)) + \sum_{i\in K, j\in \overline{K}} (-1)^{j-i} v_{S_1}[i] q_{S_2}[j-k] \det(M_i)\det(N_j)$$
$$= \det(M(S_1))\det(N(S_2)) + \left(\sum_{i\in K}(-1)^i v_{S_1}[i]\det(M_i)\right)\left(\sum_{j\in\overline{K}}(-1)^j q_{S_2}[j-k]\det(N_j)\right).$$

Similarly, using the Generalized Laplace Theorem (see [Ahm23, Theorem 3.1]), we compute B'. For each $j \in \overline{K}$, let $\widetilde{N}_j$ denote the matrix obtained by removing jth row of $N(S_2)$ and adding q_{S_2} as the first row. Then,

$$\det(B') = \det(M(S_1))\det(N(S_2)) + \left(\sum_{i\in K}(-1)^i v_{S_1}[i]\det(M_i)\right)\left(\sum_{j\in\overline{K}}(-1)^j u_{S_2}[j-k]\det(\widetilde{N}_j)\right).$$

Let P be the following $(|S_2|+1)\times(|S_2|+1)$ matrix

$$P = \begin{pmatrix} 0 & q_{S_2}^T \\ u_{S_2} & N(S_2) \end{pmatrix}.$$

Then,

$$(-1)^k \det(P) = \sum_{j\in\overline{K}}(-1)^j q_{S_2}[j-k]\det(N_j) = \sum_{j\in\overline{K}}(-1)^j u_{S_2}[j-k]\det(\widetilde{N}_j).$$

The above equalities follow from the expression for computing the determinant of P by the first row and the first column of P, respectively. Hence, $\det(A(S)) = \det(\mathrm{ct}(A,X)(S))$. □

Notations. Suppose that A is an $n\times n$ matrix over a field $\mathbb{F}$. Let $X \subseteq [n]$ such that $\mathrm{rank}(A[X,\overline{X}]) \le 1$ and $\mathrm{rank}(A[\overline{X},X]) \le 1$. Then, the matrix A has the following structure:

$$A = \begin{array}{c} \\ X \\ \\ \overline{X} \end{array}\begin{array}{c} \begin{array}{cc} X & \overline{X} \end{array} \\ \begin{pmatrix} M & p\cdot q^T \\ & \\ u\cdot v^T & N \end{pmatrix} \end{array},$$

where $p, v \in \mathbb{F}^{|X|}$ and $q, v \in \mathbb{F}^{|\overline{X}|}$. Without loss of generality, assume that $X = [\ell]$. Then, $\overline{X} = [n]\setminus[\ell]$. For each $i \in X$, let

1. M_i^C denote the $\ell\times\ell$ matrix obtained by removing ith column of M and appending p as the ℓth column.
2. M_i^R denote the $\ell\times\ell$ matrix obtained by removing ith row of M and appending v^T as the ℓth row.

For each $j \in \overline{X}$, let $\overline{X}_j$ denote the set $\overline{X} - j$. Let $p^A \in \mathbb{F}^{|X|}$ and $q^A \in \mathbb{F}^{|\overline{X}|}$ defined as follows: for all $i \in X$ and $j \in \overline{X}$,

$$p_A[i] = (-1)^{\ell+i+1}\det(M_i^C) \quad\text{and}\quad q_A[j-\ell] = \sum_{k\in\overline{X}}(-1)^{k+j} q[k-\ell]\cdot\det(N[\overline{X}_j,\overline{X}_k]). \tag{26}$$

Similarly, let $v^A \in \mathbb{F}^{|X|}$ and $u^A \in \mathbb{F}^{|\overline{X}|}$ defined as follows: for all $i \in X$ and $j \in \overline{X}$,

$$v_A[i] = (-1)^{\ell+i+1}\det(M_i^R) \quad\text{and}\quad u_A[j-\ell] = \sum_{k\in\overline{X}}(-1)^{k+j} u[k-\ell]\cdot\det(N[\overline{X}_k,\overline{X}_j]). \tag{27}$$

Based on the above notations, we have the following claim.

Claim A.1. *Considering the notations defined above,*

$$A^{\mathrm{adj}}[X, \overline{X}] = p_A \cdot q_A^T \quad and \quad A^{\mathrm{adj}}[\overline{X}, X] = u_A \cdot v_A^T,$$

where p_A, q_A, u_A and v_A are defined as Eq. (26) *and Eq.* (27).

Proof. The proof of the above claim will closely follow the proof of [Ahm23, Lemma 4.5]. For all $i \in [n]$, let $(n)_i$ denote the set $[n] \setminus \{i\}$. Then, for all $i \in X$ and $j \in \overline{X}$,

$$A^{\mathrm{adj}}[i, j] = (-1)^{i+j} \cdot \det\left(A[(n)_j, (n)_i]\right).$$

Using the Generalized Laplace Theorem (see [Ahm23, Theorem 3.1]), for all $i \in X$ and $j \in \overline{X}$,

$$\det\left(A[(n)_j, (n)_i]\right) = (-1)^{\ell-i+1} \cdot \sum_{T \subseteq [n]_i,\ |T| = \ell} (-1)^{\Sigma X + \Sigma T} \det(A[X, T]) \cdot \det(A[\overline{X}_j, \overline{T}]).$$

Since the rank of $A[X, \overline{X}]$ is at most one, for all ℓ-size subsets T of $[n]_i$ with $|T \cap \overline{X}| \geq 2$, the value of $\det(A[X, T])$ is zero. Thus, from the above equation,

$$\begin{aligned} \det\left(A[(n)_j, (n)_i]\right) &= (-1)^{\ell-i+1} \cdot \sum_{k \in \overline{X}} (-1)^{k-i} \det(M_i^C) \cdot q[k-\ell] \cdot \det(A[\overline{X}_j, \overline{X}_k]) \\ &= (-1)^{\ell+1} \det(M_i^C) \cdot \sum_{k \in \overline{X}} (-1)^k q[k-\ell] \cdot \det(A[\overline{X}_j, \overline{X}_k]) \end{aligned}$$

This implies that, for all $i \in X$ and $j \in \overline{X}$,

$$\begin{aligned} A^{\mathrm{adj}}[i, j] &= (-1)^{\ell+i+1} \det(M_i^C) \cdot \sum_{k \in \overline{X}} (-1)^{k+j} q[k-\ell] \cdot \det(A[\overline{X}_j, \overline{X}_k]) \\ &= p_A[i] \cdot q_A[j-\ell]. \end{aligned}$$

Similarly, we can show that for all $i \in X$ and $j \in \overline{X}$,

$$A^{\mathrm{adj}}[j, i] = u_A[j-\ell] \cdot v_A[i].$$

This completes the proof of the above claim. □

Lemma 2.7 (restated). *Let A be an $n \times n$ matrix over a field $\mathbb{F}$. Let D be an $n \times n$ diagonal matrix over $\mathbb{F}$ such that $A + D$ is non-singular. Then, A and $(A + D)^{\mathrm{adj}}$ have the same set of cuts.*

Proof. Observe that A and $A + D$ have the same set of cuts. Next, we show that $A + D$ and $(A + D)^{\mathrm{adj}}$ have the same set of cuts, implying that A and $(A + D)^{\mathrm{adj}}$ have the same set of cuts. From Claim A.1, any cut in $A + D$ is also a cut in $(A + D)^{\mathrm{adj}}$. For the converse direction, assume that X is a cut in $(A + D)^{\mathrm{adj}}$. Therefore, X is also a cut in $(A + D)^{-1}$ since $A + D$ is non-singular and

$$(A + D)^{-1} = \frac{1}{\det(A + D)} (A + D)^{\mathrm{adj}}.$$

Note that

$$A + D = \det(A + D) \cdot ((A + D)^{-1})^{\mathrm{adj}}.$$

Thus, again using Claim A.1, X is also a cut in $A + D$. □

Lemma 2.15 (restated). *Let A be an $n \times n$ matrix over a field $\mathbb{F}$. Then, for any $X \subseteq [n]$ with* $\mathrm{rank}(A[X, \overline{X}]) \leq 1$ *and* $\mathrm{rank}(A[\overline{X}, X]) \leq 1$,

$$\mathrm{ct}(A, X)^{\mathrm{adj}} = \mathrm{ct}(A^{\mathrm{adj}}, X).$$

Proof. Without loss of generality, assume that $X = [\ell]$. Then, $\overline{X} = [n] \setminus [\ell]$. Note that A can be written as follows.

$$A = \begin{array}{c} \\ X \\ \overline{X} \end{array} \begin{array}{c} \begin{array}{cc} X & \overline{X} \end{array} \\ \begin{pmatrix} M & p \cdot q^T \\ u \cdot v^T & N \end{pmatrix} \end{array},$$

where $p, v \in \mathbb{F}^{|X|}$ and $q, u \in \mathbb{F}^{|\overline{X}|}$. From Definition 2.11,

$$\widetilde{A} = \mathrm{ct}(A, X) = \begin{array}{c} \\ X \\ \overline{X} \end{array} \begin{array}{c} \begin{array}{cc} X & \overline{X} \end{array} \\ \begin{pmatrix} M & p \cdot u^T \\ q \cdot v^T & N^T \end{pmatrix} \end{array}.$$

Repeating some notations from the above, for each $i \in X$, let

1. M_i^C denote the $\ell \times \ell$ matrix obtained by removing ith column of M and appending p as the ℓth column.
2. M_i^R denote the $\ell \times \ell$ matrix obtained by removing ith row of M and appending v^T as the ℓth row.

For each $j \in \overline{X}$, let $\overline{X}_j$ denote the set $\overline{X} - j$. Then, using Claim A.1,

$$A^{\mathrm{adj}} = \begin{array}{c} \\ X \\ \overline{X} \end{array} \begin{array}{c} \begin{array}{cc} X & \overline{X} \end{array} \\ \begin{pmatrix} A^{\mathrm{adj}}(X) & p_A \cdot q_A^T \\ u_A \cdot v_A^T & A^{\mathrm{adj}}(\overline{X}) \end{pmatrix} \end{array},$$

where $p_A \in \mathbb{F}^{|X|}$ and $q_A \in \mathbb{F}^{|\overline{X}|}$ are defined as Eq. (26) and $v_A \in \mathbb{F}^{|X|}$ and $u_A \in \mathbb{F}^{|\overline{X}|}$ are defined as Eq. (27). Then,

$$\mathrm{ct}(A^{\mathrm{adj}}, X) = \begin{array}{c} \\ X \\ \overline{X} \end{array} \begin{array}{c} \begin{array}{cc} X & \overline{X} \end{array} \\ \begin{pmatrix} A^{\mathrm{adj}}(X) & p_A \cdot u_A^T \\ q_A \cdot v_A^T & A^{\mathrm{adj}}(\overline{X})^T \end{pmatrix} \end{array}.$$

On the other hand, again applying Claim A.1,

$$\widetilde{A}^{\mathrm{adj}} = \begin{array}{c} \\ X \\ \overline{X} \end{array} \begin{array}{c} \begin{array}{cc} X & \overline{X} \end{array} \\ \begin{pmatrix} \widetilde{A}^{\mathrm{adj}}(X) & \tilde{p} \cdot \tilde{u}^T \\ \tilde{q} \cdot \tilde{v}^T & \widetilde{A}^{\mathrm{adj}}(\overline{X}) \end{pmatrix} \end{array},$$

where like Eq. (26) and Eq. (27), $\tilde{p}, \tilde{v} \in \mathbb{F}^{|X|}$ and $\tilde{q}, \tilde{u} \in \mathbb{F}^{|\overline{X}|}$ are defined as follows: For $i \in X$ and $j \in \overline{X}$,

$$\tilde{p}[i] = (-1)^{\ell+i+1} \det(M_i^C) \quad \text{and} \quad \tilde{u}[j-\ell] = \sum_{k \in \overline{X}} (-1)^{k+j} u[k-\ell] \cdot \det(N^T[\overline{X}_j, \overline{X}_k])$$
$$\tilde{v}[i] = (-1)^{\ell+i+1} \det(M_i^R) \quad \text{and} \quad \tilde{q}[j-\ell] = \sum_{k \in \overline{X}} (-1)^{k+j} q[k-\ell] \cdot \det(N^T[\overline{X}_k, \overline{X}_j]).$$

Now we show that $\mathrm{ct}(A^{\mathrm{adj}}, X) = \widetilde{A}^{\mathrm{adj}}$. For all $i \in [n]$, let $(n)_i$ denote the set $[n] \setminus \{i\}$. Next, we divide our proof into three cases.

1. Assume that $i, j \in X$. Then,

$$\mathrm{ct}(A^{\mathrm{adj}}, X)[i,j] = A^{\mathrm{adj}}[i,j] = \det(A[(n)_j, (n)_i]).$$

On the other hand,

$$\widetilde{A}^{\mathrm{adj}}[i,j] = \det(\widetilde{A}[(n)_j, (n)_i]) = \det(\mathrm{ct}(A[(n)_j, (n)_i], \overline{X})).$$

Therefore, applying Lemma 2.14, $\mathrm{ct}(A^{\mathrm{adj}}, X)[i,j] = \widetilde{A}^{\mathrm{adj}}[i,j]$ for all $i, j \in X$.

2. Assume that $i, j \in \overline{X}$. Then,

$$\mathrm{ct}(A^{\mathrm{adj}}, X)[i,j] = A^{\mathrm{adj}}[j,i] = \det(A[(n)_i, (n)_j]).$$

On the other hand,

$$\widetilde{A}^{\mathrm{adj}}[i,j] = \det(\widetilde{A}[(n)_j, (n)_i]) = \det(\mathrm{ct}(A[(n)_i, (n)_j], X)).$$

Therefore, again applying Lemma 2.14, $\mathrm{ct}(A^{\mathrm{adj}}, X)[i,j] = \widetilde{A}^{\mathrm{adj}}[i,j]$ for all $i, j \in \overline{X}$.

3. Assume that $i \in X$ and $j \in \overline{X}$. Then,

$$\begin{aligned}
\mathrm{ct}(A^{\mathrm{adj}}, X) &= p_A[i] \cdot u_A[j-\ell] \\
&= (-1)^{\ell+i+1} \det(M_i^C) \cdot \sum_{k \in \overline{X}} (-1)^{k+j} u[k-\ell] \cdot \det(N[\overline{X}_k, \overline{X}_j]) \\
&= (-1)^{\ell+i+1} \det(M_i^C) \cdot \sum_{k \in \overline{X}} (-1)^{k+j} u[k-\ell] \cdot \det(N^T[\overline{X}_j, \overline{X}_k]) \\
&= \tilde{p}[i] \cdot \tilde{u}[j] = \widetilde{A}^{\mathrm{adj}}[i,j].
\end{aligned}$$

Similarly, we can show that $\mathrm{ct}(A^{\mathrm{adj}}, X)[j,i] = \widetilde{A}^{\mathrm{adj}}[j,i]$.

This completes the proof of our lemma. □

B Others

For an $n \times n$ matrix A, let G_A be the graph defined **??**. *For a directed cycle C of G_A, let the weight of the cycle denoted by $w_A(C)$ be $\prod_{(i,j) \in C} A[i,j]$.*

Lemma B.1. *Let A and B be two $n \times n$ matrices. Then $A \stackrel{\text{PME}}{=} B$ if and only if for each subset $S \subseteq [n]$, the sum of weights of directed Hamiltonian cycles is the same for subgraphs $G_A[S]$ and $G_B[S]$.*

Proof. We show this by induction on the size of subsets. The base case, when the size of the subset is one, is trivial. Suppose the statement is true for subsets of size k. Let S be a subset of size $k+1$ and $\det(A(\emptyset)) = \det(B(\emptyset)) = 1$ and $\mathcal{C}_A$ and $\mathcal{C}_B$ denote the set of Hamiltonian cycles of $G_A[S]$ and $G_B[S]$, respectively. Then,

$$\det(A(S)) = \sum_{T \neq \emptyset, T \subseteq S} \pm(\det(A(S \setminus T)) \prod_{i \in T} A(i)) \pm \sum_{C \in \mathcal{C}_A} w_A(C). \tag{28}$$

$$\det(B(S)) = \sum_{T \neq \emptyset, T \subseteq S} \pm(\det(B(S \setminus T)) \prod_{i \in T} B(i)) \pm \sum_{C' \in \mathcal{C}_B} w_B(C'). \tag{29}$$

The backward direction follows directly from Eqs. (28) and (29) as all the principal minors of A and B are the same, and the signs of corresponding summands are the same in Eqs. (28) and (29).

Now, we show the forward direction. For any non-empty subset T, if we consider the submatrices $A(S \setminus T)$ and $B(S \setminus T)$, for each subset $T' \subseteq S \setminus T$ the sum of the weights of the directed Hamiltonian cycles in $G_A[T']$ and $G_B[T']$ are same. Hence, $\det(A(S \setminus T)) = \det(B(S \setminus T))$ by induction hypothesis. Also, the signs are the same in Eq. (28) and Eq. (29) as it just depends on the size of the subsets T. This, along with the fact that the sum of weights of the Hamiltonian cycle in $G_A[S]$ and $G_B[S]$ are the same, implies $\det(A(S)) = \det(B(S))$.

$\square$

Following is an immediate corollary of the above lemma.

Corollary B.2. *Let A and B be two $n \times n$ matrices such that $A \stackrel{\text{PME}}{=} B$. Let D be an $n \times n$ diagonal matrix. Then, $A + D \stackrel{\text{PME}}{=} B + D$.*

Proof. Since $A \stackrel{\text{PME}}{=} B$, $(A + D)[i] = (B + D)[i]$ for each $i \in [n]$. This implies the sum of weights of the Hamiltonian cycle in $G_A[i]$ and $G_B[i]$ are the same. Since $A \stackrel{\text{PME}}{=} B$, from Lemma B.1, for each subset $S \subseteq [n]$, the sum of weights of directed Hamiltonian cycles is the same for subgraphs $G_A[S]$ and $G_B[S]$. Note that G_A and G_{A+D} have the same set of cycles, and any cycle of length greater than one has the same weight in G_A and G_{A+D}. Similarly, G_B and G_{B+D} have the same set of cycles, and any cycle of length greater than one has the same weight in G_B and G_{B+D}. This implies that for each subset $S \subseteq [n]$, the sum of weights of directed Hamiltonian cycles is the same for subgraphs $G_{A+D}[S]$ and $G_{B+D}[S]$. Hence, from Lemma B.1, $A + D \stackrel{\text{PME}}{=} B + D$. $\square$

Claim B.3. *Let A be a 4×4 matrix with all off-diagonal entries as non-zero. Let B be another 4×4 matrix such that $A \stackrel{\text{PME}}{=} B$. Then, one of the following holds:*

1. *$A \stackrel{DS}{=} B$ or $A \stackrel{DS}{=} B^T$*
2. *A and B has a common cut and for any common cut X of A and B, $\mathrm{ct}(A, X) \stackrel{DS}{=} B$ or $\mathrm{ct}(A, X) \stackrel{DS}{=} B^T$*

Proof. When A does not have any cut, then from Lemma 2.9 $A \stackrel{\text{DS}}{=} B$ or $A \stackrel{\text{DS}}{=} B^T$. Suppose A has a cut. Then, B must have some cut; otherwise, since $A \stackrel{\text{PME}}{=} B$, Lemma 2.9 would imply that A has no cut, which is a contradiction. First, we show that B must have a cut that is common to A using contradiction. Suppose this is not true. Without loss of generality, assume that A has cut $\{1, 2\}$ and B has cut $\{1, 3\}$. Since $A \stackrel{\text{PME}}{=} B$, $A' = D_1 A D_1^{-1} \stackrel{\text{PME}}{=} D_2 B D_2^{-1} = B'$ for any non-singular diagonal matrices D_1 and D_2. Since off-diagonal entries are non-zero, we can choose D_1 such that $A'[1, 3] = A'[1, 4] = A'[2, 3] = 1$. Since A and A' has same cuts, $\mathrm{rank}(A'[\{1, 2\}, \{3, 4\}]) = 1$ and hence $A'[2, 4] = 1$. If the above claim is true for A' and B then it is also true for A and B. Hence, without loss of generality, we can assume that each entry of $A[\{1, 2\}, \{3, 4\}]$ is one. Similarly, we can assume that each entry of $B[\{1, 3\}, \{2, 4\}]$ is one. Let A be the following matrix with non-zero off-diagonal entries.

$$A = \begin{pmatrix} * & a & 1 & 1 \\ b & * & 1 & 1 \\ c & dc & * & e \\ f & df & g & * \end{pmatrix}$$

Since $A \stackrel{\text{PME}}{=} B$, we can represent B as follows by making its size two principal minors the same as of A.

$$B = \begin{pmatrix} * & 1 & h & 1 \\ ab & * & dc & i \\ \frac{c}{h} & 1 & * & 1 \\ f & \frac{df}{i} & eg & * \end{pmatrix}$$

Since $\text{rank}(B[\{2,4\},\{1,3\}]) = 1$, $abeg = dcf$. After substituting g with $\frac{dcf}{abe}$,

$$A = \begin{pmatrix} * & a & 1 & 1 \\ b & * & 1 & 1 \\ c & dc & * & e \\ f & df & \frac{dcf}{abe} & * \end{pmatrix} \text{ and } B = \begin{pmatrix} * & 1 & h & 1 \\ ab & * & dc & i \\ \frac{c}{h} & 1 & * & 1 \\ f & \frac{df}{i} & \frac{dcf}{ab} & * \end{pmatrix}.$$

Since $A \stackrel{\text{PME}}{=} B$, from Lemma B.1, we get the following equations by equating the sum of weights of Hamiltonian cycles in $G_A[S]$ and $G_B[S]$ such that $|S| = 3$.

$$ac + bdc = \frac{dc^2}{h} + abh \implies \left(a - \frac{dc}{h}\right)(c - bh) = 0 \qquad [S = \{1,2,3\}] \tag{30}$$

$$af + bdf = if + \frac{abdf}{i} \implies f(a-i)\left(1 - \frac{bd}{i}\right) = 0 \qquad [S = \{1,2,4\}] \tag{31}$$

$$ef + \frac{dc^2 f}{abe} = hf + \frac{dc^2 f}{abh} \implies f(e-h)\left(1 - \frac{dc^2}{abeh}\right) = 0 \qquad [S = \{1,3,4\}] \tag{32}$$

$$edf + \frac{d^2c^2 f}{abe} = \frac{d^2 cf}{i} + \frac{dcif}{ab} \implies fd\left(e - \frac{cd}{i}\right)\left(1 - \frac{ci}{abe}\right) = 0 \qquad [S = \{2,3,4\}]. \tag{33}$$

Since the off-diagonal entries are non-zero, in each equation, at least one of the last two factors must be zero. This gives 16 different possibilities because there are four equations. Now, we show that each of these possibilities would imply a contradiction.

Note that $\left(a - \frac{dc}{h}\right) = 0$ and $(e - h) = 0$ together implies $ae = dc$ which in turn implies $\{1,3\}$ is a cut of A which is a contradiction. Hence, $\left(a - \frac{dc}{h}\right)$ and $e - h$ can't be zero together. Similarly, $\left(a - \frac{dc}{h}\right)$ and $a - i$ can't be zero together as it implies $cd = hi$. This implies $\{1,2\}$ is a cut of B, which contradicts the condition of no common cut. Hence, if $\left(a - \frac{dc}{h}\right) = 0$ then $\left(1 - \frac{bd}{i}\right)$ and $\left(1 - \frac{dc^2}{abeh}\right)$ must be zero to make Eqs. (31) and (32) zero. $\left(a - \frac{dc}{h}\right) = 0$ and $\left(1 - \frac{bd}{i}\right) = 0$ imply $abh = ci$ which in turn implies $\{1,4\}$ is a cut of B. Similarly, $\left(a - \frac{dc}{h}\right) = 0$ and $\left(1 - \frac{dc^2}{abeh}\right) = 0$ imply $c = be$. This implies $\{1,4\}$ is also a cut of A, which contradicts the no common cut condition. Note that $\left(a - \frac{dc}{h}\right) = 0$ always led to a contradiction. Hence, it must be that $(c - bh) = 0$ so that Eq. (30) is satisfied.

Now, we show that $(c - bh) = 0$ would also always lead to a contradiction. If $\left(1 - \frac{bd}{i}\right)$ is also zero, then $cd = hi$ which implies $\{1,2\}$ is a cut of B which is a contradiction. Similarly, if $\left(1 - \frac{dc^2}{abeh}\right) = 0$ then $cd = ae$ which implies $\{1,3\}$ is a cut of A which contradicts the no common cut condition. Hence, to satisfy Eqs. (31) and (32), it must be that $(e - h)$ and $(a - i)$ is equal to zero along with $(c - bh)$. However, then $\{1,4\}$ becomes a cut of both A and B. Hence, $(c - bh)$ also can't be zero. This contradicts $Eq.$ (30). Hence, if A has a cut and $A \stackrel{\text{PME}}{=} B$, then A and B must have a common cut.

Without loss of generality, let that common cut be $\{1,2\}$. Using earlier arguments, we can represent A and B as follows by making all entries of $A[\{1,2\},\{3,4\}]$ and $B[\{1,2\},\{3,4\}]$ one and equating size two principal minors.

$$A = \begin{pmatrix} * & a & 1 & 1 \\ b & * & 1 & 1 \\ c & dc & * & e \\ f & df & g & * \end{pmatrix} \text{ and } B = \begin{pmatrix} * & h & 1 & 1 \\ \frac{ab}{h} & * & 1 & 1 \\ c & dc & * & i \\ f & df & \frac{ge}{i} & * \end{pmatrix}.$$

Since $A \stackrel{\text{PME}}{=} B$, from Lemma B.1, we get the following equations by equating the sum of weights of Hamiltonian cycles in $G_A[S]$ and $G_B[S]$ for $S = \{1,2,3\}$ and $S = \{1,3,4\}$.

$$ac + bdc = hc + \frac{abdc}{h} \implies c(a-h)\left(1 - \frac{bd}{h}\right) = 0 \qquad [S = \{1,2,3\}] \tag{34}$$

$$ef + cg = if + \frac{cge}{i} \implies (e-i)\left(f - \frac{cg}{i}\right) = 0 \qquad [S = \{1,3,4\}] \tag{35}$$

Eqs. (34) and (35) together implies the following four possible cases. $(a-h) = 0$ and $(e-i) = 0$ implies $A = B$. Hence, $A \stackrel{\text{DS}}{=} B$.

If $\left(1 - \frac{bd}{h}\right) = 0$ and $\left(f - \frac{cg}{i}\right) = 0$, then

$$B = \begin{pmatrix} * & bd & 1 & 1 \\ \frac{a}{d} & * & 1 & 1 \\ c & dc & * & \frac{cg}{f} \\ f & df & \frac{fe}{c} & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{d} & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & f \end{pmatrix} \begin{pmatrix} * & b & c & f \\ a & * & dc & df \\ 1 & 1 & * & g \\ 1 & 1 & e & * \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & \frac{1}{c} & 0 \\ 0 & 0 & 0 & \frac{1}{f} \end{pmatrix} = DA^TD^{-1}.$$

Hence, in this case, we get $A \stackrel{\text{DS}}{=} B^T$.

If $(a-h) = 0$ and $\left(f - \frac{cg}{i}\right) = 0$, then

$$B = \begin{pmatrix} * & a & 1 & 1 \\ b & * & 1 & 1 \\ c & dc & * & \frac{cg}{f} \\ f & df & \frac{fe}{c} & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & f \end{pmatrix} \begin{pmatrix} * & a & c & f \\ b & * & c & f \\ 1 & d & * & g \\ 1 & d & e & * \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{c} & 0 \\ 0 & 0 & 0 & \frac{1}{f} \end{pmatrix}.$$

Here, the matrix in the middle on the right hand side is $\text{ct}(A, \{1,2\})$. Hence, in this case, $B \stackrel{\text{DS}}{=} \text{ct}(A, S)$.

Finally, the last case where $\left(1 - \frac{bd}{h}\right)$ and $e - i$ are zero. Then,

$$B^T = \begin{pmatrix} * & \frac{a}{d} & c & f \\ bd & * & dc & df \\ 1 & 1 & * & g \\ 1 & 1 & e & * \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} * & a & c & f \\ b & * & c & f \\ 1 & d & * & g \\ 1 & d & e & * \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{d} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Like the previous case, the matrix in the middle is $\text{ct}(A, \{1,2\})$. Hence, in this case, $\text{ct}(A, S) \stackrel{\text{DS}}{=} B^T$. This completes the proof of Claim B.3. □